CONFORMAL INVARIANTS CORRESPONDING TO PAIRS OF DOMAINS

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Abstract. We construct conformally invariant domain functions, using invariant derivatives invented by Minda. These domain functions contain information about how one domain is embedded in another. Using the Dirichlet principle and reproducing formulas for the invariant derivatives, inequalities are derived for the domain functions, in the cases that the outer domain is equivalent to the disc, to the plane or to the Riemann sphere.

1. Introduction

A general problem in univalent function theory is to describe a set of mappings from one fixed domain into another - for example, by estimating the derivatives of different orders. Looking at this problem from an 'intrinsic' point of view, one could try to describe rather the class of image domains inside the target domain. In this point of view, inequalities are given in terms of intrinsic domain functions such as Green's function, the Bergman kernel, or the hyperbolic metric. In the case of simply connected domains, one can recover inequalities for the mapping function, since the domain functions have simple relations with the mapping function.

Another way of phrasing this general problem, then, is as follows: 1) identify quantities that express how one domain sits in another, complex analytically, and 2) identify restrictions on these quantities. In order to make this concrete, consider the well-known inequality

$$\sum_{\mu,\nu} \alpha_\mu \alpha_\nu \left(g_1(\zeta_\mu, \zeta_\nu) - g_2(\zeta_\mu, \zeta_\nu) \right) \geq 0, \tag{1}$$

where g_i are Green's functions of domains $D_1 \subset D_2$, α_μ are real parameters, and $\zeta_\mu \in D_1$. The quantity on the left-hand side contains information on how D_1 sits in D_2, and the inequality is a restriction that holds for any such pair of domains (a consequence of the maximum principle.) Setting D_2 to be the unit disc in (1), we can derive inequalities for a univalent mapping function from the disc onto D_1 (for example the Schwarz lemma.)

There are two requirements for answers to this problem that will be demanded here. The first requirement is that, since this is after all complex analysis, the quantities of part 1) of the problem should represent conformal geometry; thus we demand that the quantities be conformally invariant. (There is a loose analogy here with the second fundamental form, which expresses how one Riemannian manifold sits in another.) An example of this kind of conformal invariant is given by the quantity on the left-hand side of (1).

The second requirement is that the inequalities derived should hold whether the outer domain is equivalent to the disc, to the complex plane or to the sphere. For the most part this is accomplished; though in the case of the sphere, an extra condition is required on the domain in order to say anything at all. We choose the condition that the inner domain be 'elliptically schlicht'. The meaning of this condition is explained in Section 2.2.

The main results are conformally invariant inequalities, involving Green's function and its derivatives for a pair of domains (Theorem 1). These inequalities are stated and proven in Section 4. The two tools required are the Dirichlet principle, following a method of Nehari [12], and a generalization of the Cauchy kernel to constant curvature metrics. The generalization follows quite naturally by considering the dependence of the 'mean value property' on the metric.

In Section 3, the notion of conformal invariance is made precise in the context of the main problem. In order to construct conformal invariants, and formulate the main theorem, some tools from Riemannian geometry are necessary, namely the derivatives of Minda [9] and Peschl [15]. These are described in Section 2.

2. Geometric preliminaries

2.1. **Invariant derivatives.** In order to continue, we need a special kind of differentiation, which comes from a compromise between Riemannian and conformal geometry (see Remark 1 below). This derivative was first defined by Peschl [15] and later generalized to arbitrary conformal metrics by Minda [9]. These derivatives are apparently also used in physics (see Nakahara [11] 14.1.)

We now define the covariant derivatives of Minda. The Riemannian metrics considered will be compatible with the complex structure; i.e. we consider only metrics that can be represented with a line element $\rho(z)|dz|$. Let

$$\Gamma_\rho = 2\frac{\partial}{\partial z}\log\rho \tag{2}$$

be the Christoffel symbol of $\rho|dz|$. We define two derivatives ∇_ρ and $\overline{\nabla}_\rho$ which act on tensors of the form

$$g(z)dz^n \odot d\bar{z}^m$$

(where $\odot$ denotes the symmetric tensor product) via the rule

$$\begin{aligned} \nabla_\rho g(z)dz^n \odot d\bar{z}^m &= \left(\frac{\partial g}{\partial z} - n\Gamma_\rho g\right) dz^{n+1} \odot d\bar{z}^m \\ \overline{\nabla}_\rho g(z)dz^n \odot d\bar{z}^m &= \left(\frac{\partial g}{\partial \bar{z}} - m\overline{\Gamma}_\rho g\right) dz^n \odot d\bar{z}^{m+1}. \end{aligned} \tag{3}$$

It is easily checked that ∇_ρ and $\overline{\nabla}_\rho$ satisfy a Leibniz rule with respect to the multiplication

$$g_1 dz^{n_1} \odot d\bar{z}^{m_1} \times g_2 dz^{n_2} \odot d\bar{z}^{m_2} \longmapsto g_1 g_2 dz^{n_1+n_2} \odot d\bar{z}^{m_1+m_2}.$$

Remark 1. ∇ and $\overline{\nabla}$ are related to the ordinary Riemannian connection ∇^R. Without going into the necessary identifications between complex and real objects, the relation is essentially $\nabla^R = \nabla + \overline{\nabla}$.

For holomorphic functions f, Minda [9] defines derivatives '$D^n_\rho f$' inductively by a rule equivalent to (3); explicitly

$$\nabla^n_\rho f(z) = \rho^n(z) D^n_\rho f(z) dz^n.$$

In the particular case that $\rho \equiv 1$ (i.e. if ρ is the Euclidean metric), $D^n_\rho f(z) = f^{(n)} dz^n$. For applications involving special cases of ρ, see for instance Ma and Minda [7], [8].

Remark 2. Minda defines the derivatives in greater generality, to allow for a metric other than the Euclidean on the image; what we've given here is the case that the image metric is the Euclidean. In the general case the nth derivative is a tensor of the form

$$g(z) \frac{\partial}{\partial w} \odot dz^n$$

where $w = f(z)$ (see [16]). (Since $\nabla(\partial/\partial w) = 0$ in the Euclidean metric, we can ignore this factor.)

2.2. **Elliptically schlicht functions and constant curvature metrics.** As mentioned in the introduction, it is desirable that the inequalities derived should hold whether the outer domain is equivalent to the disc, to the plane, or to the Riemann sphere. The natural metrics on each of these domains are certain constant curvature metrics. Here, we define these metrics, which will be used to state the main inequalities. We also define a special class of domains in the sphere defined by Grunsky [5] called 'elliptically schlicht' domains. See also for instance Kühnau [6] or Duren and Kühnau [4].

Consider the conformal metric $\lambda_k(z)^2 |dz|^2$, where

$$\lambda_k(z) = \frac{\sqrt{|k|}}{1 + k|z|^2}.$$

This metric is defined on the disc of radius $1/\sqrt{|k|}$ if $k < 0$, and defined on the whole sphere if $k > 0$. Set

$$\lambda_0(z) = 1$$

in the case that $k = 0$. We will often abuse notation and also refer to the function λ_k a 'metric'. This metric has curvature 4, 0, and -4, in the cases that k is positive, zero, or negative respectively. For $k < 0$, λ_k is the unique complete constant curvature metric on the disc of radius $1/\sqrt{|k|}$ (up to scale). On the sphere, there are many complete constant positive curvature metrics; λ_k are the only ones such that 0 and ∞ are antipodal (up to scale). The isometries of λ_k are the mappings of the form

$$T(z) = e^{i\theta} \frac{z - a}{1 + k\bar{a}z}.$$

The points w and $-1/k\bar{w}$ are antipodal in the metric λ_k. simply connected domains that contain no pairs of points w and $-1/k\bar{w}$ will be called '**elliptically k-schlicht**'. Univalent mappings onto such domains will also be referred to as 'elliptically k-schlicht'. Setting $k = 1$ recovers the standard definition of Grunsky.

If $k < 0$, the point $-1/(k\bar{w})$ is the reflection of w in the circle of radius $1/\sqrt{|k|}$; a domain not containing any pairs of reflected points must be bounded in the disc of radius $1/\sqrt{|k|}$ or its complement. Thus we see that the condition of being 'elliptical k-schlicht' is in a sense analogous to the condition of boundedness.

Note that with the parameter k, one can recover the case of the plane: as $k \to 0^+$, the condition of elliptic k-schlichtness becomes the requirement that the domain not contain ∞; similarly as $k \to 0^-$. The parameter k allows one to continuously interpolate between the hyperbolic, planar, and elliptic cases of inequalities.

For more discussion of the notion of elliptically schlicht domains and of results by other authors see [17].

2.3. **Domain functions.** In this section we define some domain functions for which inequalities will later be derived.

Let D be a domain in the Riemann sphere possessing a Green's function g. Define the Bergman kernel of D to be

$$K(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g}{\partial\zeta\partial\overline{\eta}}(\zeta,\eta). \tag{4}$$

We also have the (apparently nameless) kernel function (Bergman and Schiffer [2])

$$L(\zeta,\eta) = -\frac{2}{\pi}\frac{\partial^2 g}{\partial\zeta\partial\eta}(\zeta,\eta). \tag{5}$$

We also need versions of these functions on the plane and sphere. On the sphere, we have a kind of 'Green's function'

$$g_k(\zeta,\eta) = -\log\left|\frac{\sqrt{|k|}(\zeta-\eta)}{(1+k\overline{\eta}\zeta)}\right|. \tag{6}$$

This is the unique harmonic function (up to an additive constant) with opposite logarithmic singularities at the antipodal points η and $-1/k\overline{\eta}$; this 'Green's function' depends on the choice of constant curvature metric λ_k (because this choice determines which points are antipodal.) Note that if we take k to be negative, the formula above agrees with Green's function on the disc of radius $1/\sqrt{|k|}$.

Using this definition of Green's function, we can also extend the kernels (4) and (5) to the whole sphere, simply by replacing 'g' with 'g_k'. Explicitly,

$$K_k(\zeta,\eta) = -\frac{1}{\pi}\frac{k}{(1+k\bar{\zeta}\eta)^2} \tag{7}$$

and

$$L_k(\zeta,\eta) = \frac{1}{\pi(\zeta-\eta)^2}. \tag{8}$$

Again, both definitions agree with the standard ones on the disc of radius $1/\sqrt{|k|}$ when $k<0$. Finally, note that we can extend these definitions to the plane by setting $k=0$.

2.4. **Kernels which reproduce a holomorphic function and its invariant derivatives.** In this section we construct kernels which, when integrated against a holomorphic function around a simple closed curve, reproduce the function. These kernels depend on the choice of metric λ_k, and arise naturally from geometric considerations. For negative curvature, on simply connected domains, these geometric considerations just amount to an interpretation of the function $\partial g/\partial z$ in terms of hyperbolic angle. The reproducing kernels provide an elegant counterpart to the Cauchy kernel for the more general metrics, and appear naturally in the proof of the main inequalities in Theorem 1. Pressing the analogy further, we give kernels which reproduce the derivatives of Minda; the result is a generalization of the Cauchy formula for derivatives.

The relation between the Cauchy kernel and the mean value property is of course well-known; if one restricts attention to a Euclidean circle centred at the point a, then, denoting the infinitesimal angle traced around this circle by $d\theta_a$, we have

$$d\theta_a = \frac{1}{i}\frac{dz}{z-a}.$$

So the reproducing property of the Cauchy kernel is the same as the mean value property of harmonic functions, at least around a circle centred at the point in question. Once we recognize that the kernel is holomorphic, it is seen that the right-hand side functions as a reproducing kernel for holomorphic functions, around an arbitrary smooth simple closed curve homotopic to the point a.

We now construct the angle element corresponding to the metric λ_k; from this it is possible to generalize the Cauchy kernel. Consider a λ_k-circle γ centred at the point a, in either the disc of radius $1/\sqrt{|k|}$, $\mathbb{C}$ or $\overline{\mathbb{C}}$, in the cases that $k<0$, $k=0$, and $k>0$ respectively. Let

$$T(z) = \frac{z-a}{1+k\bar{a}z}.$$

This is an isometry of the metric λ_k, so, denoting the angle element around a by $d\theta_a^k$,

$$d\theta_a^k = T^*(d\theta_0^k).$$

Since $T\circ\gamma$ is a Euclidean circle centred at 0, by the radial symmetry of λ_k, we have that along $T\circ\gamma$,

$$d\theta_0^k = \frac{1}{i}\frac{dz}{z}.$$

So

$$\begin{aligned} d\theta_a^k &= T^*\left(\frac{1}{i}\frac{dz}{z}\right) = \frac{1}{i}\frac{T'(z)}{T(z)}dz \\ &= \frac{1}{i}\frac{1+k|a|^2}{(z-a)(1+k\bar{a}z)}dz \end{aligned} \tag{9}$$

Note that this is real by construction.

One can then compute, either explicitly or using a change of variables, that for a λ_k-circle γ and any function u harmonic on a region containing γ and its interior,

$$u(a) = \frac{1}{2\pi}\int_\gamma u(z)d\theta_a^k.$$

This is the analogue of the mean value property, for the metrics λ_k.

Remark 3. The Laplacian depends on the choice of metric. However, in two dimensions, the Laplacian of a conformal metric $\rho(z)|dz|$ is $\rho^{-2}\Delta$ where Δ is the standard Laplacian. Thus, the notion of harmonic function does not depend on the metric and agrees with the classical one.

Since $d\theta_a^k$ is holomorphic, we have that for any smooth simple closed curve γ homotopic to a, and f holomorphic on a region containing γ and its interior,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{1+k|a|^2}{(z-a)(1+k\bar{a}z)}f(z)dz \tag{10}$$

(this can of course be computed directly.) In the case that $k>0$, it is necessary to assume that γ does not contain the antipodal point $-1/(k\bar{a})$.

In order to produce derivatives of a holomorphic function, one differentiates under the integral sign in the Cauchy formula, to get the Cauchy formula for derivatives

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz.$$

To get rather invariant derivatives of a holomorphic function, one differentiates under the integral sign with the connection ∇. This leads to the following.

Proposition 1. *Let ∇ be the connection corresponding to λ_k on the domain D_k, $\mathbb{C}$, or $\overline{\mathbb{C}}$ in the case that $k<0$, $k=0$, and $k>0$ respectively. Let γ be a simple closed smooth curve in the plane, enclosing the point w. Let f be a function holomorphic on a region containing γ and its interior. In the case $k>0$, assume also that γ does not enclose $-1/(k\bar{w})$. Then,*

$$\nabla^n f(w) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)(1+k\bar{w}z)^{n-1}}{(1+k|z|^2)^{n-1}(z-w)^{n+1}}dz\cdot dw^n.$$

Proof. Differentiate under the integral sign using equation (10) and the rule (3). □

Note that setting $k=0$ recovers the Cauchy formulas for derivatives.

We can easily compute, that for the Green's function of the disc of radius $1/\sqrt{|k|}$ (given by equation (6)),

$$g_k(z,a) = -\log\left|\frac{\sqrt{|k|}(z-a)}{1+k\bar{a}z}\right|$$

that

$$\frac{\partial g_k}{\partial z} = -\frac{1}{2}\frac{1+k|a|^2}{(z-a)(1+k\bar{a}z)}$$

so

$$d\theta_a^k = -\frac{2}{i}\frac{\partial g_k}{\partial z}(z,a)dz.$$

Of course, it is true in general that for a smooth null-homotopic curve containing a in any domain D possessing a Green's function g, and f holomorphic in a region containing γ and its interior,

$$f(a) = -\frac{1}{\pi i}\int_\gamma \frac{\partial g}{\partial z}(z,a)f(a)dz,$$

so (10) is just a reinterpretation of this well-known property of Green's function in the simply connected negative curvature case. (It follows from conformal invariance of both hyperbolic angle and Green's function that

$$-\frac{2}{i}\frac{\partial g}{\partial z}dz$$

is hyperbolic angle on any simply connected domain equivalent to the disc.)

To prove the main theorem, we need a slightly different kernel which reproduces derivatives of harmonic functions.

Proposition 2. *Let $D_1 \subset D_2$, where D_2 is a simply connected domain. Let ∇ be the connection corresponding to the hyperbolic metric, Euclidean metric, or λ_k, $k>0$ on D_2, in the cases that D_2 is equivalent to the disc, plane, and sphere respectively. Let g_1 be Green's function of D_1. If $D_2=\overline{\mathbb{C}}$, we assume further that*

D_1 is elliptically k-schlicht. Then, for all functions h which are harmonic on D_1 and extend continuously to the boundary, for any $\zeta_0 \in D_1$,

$$-\frac{1}{\pi i}\int_{\partial D_1} \overline{\nabla}_\eta^n\Big|_{\eta=\zeta_0} g_{1\,\zeta}(\zeta,\eta)h(\zeta)d\zeta = \overline{\nabla}^n h(\zeta_0)$$

and

$$-\frac{1}{\pi i}\int_{\partial D_1} \nabla_\eta^n\big|_{\eta=\zeta_0} g_{1\,\zeta}(\zeta,\eta)h(\zeta)d\zeta = \nabla^n h(\zeta_0)$$

where ∇_η denotes differentiation in the variable η.

Proof. On ∂D_1,

$$\frac{2}{i}\frac{\partial g_1}{\partial z}dz = \frac{\partial g_1}{\partial n}ds$$

where n is the outward unit normal in z, so by Green's identity, for any harmonic function h on D_1,

$$h(a) = -\frac{1}{\pi i}\int_{\partial D_1} \frac{\partial g_1}{\partial z}(z,a)h(z)dz.$$

Now differentiate under the integral sign. □

3. Conformal invariance

In this section we introduce a notion of conformal invariance. We consider configurations consisting of a pair of domains D_1 and D_2 such that $D_1 \subset D_2$. We further assume that D_1 and D_2 both possess Green's functions g_1, g_2 and complete constant negative curvature metrics $\lambda_1(z)|dz|$, $\lambda_2(z)|dz|$. Represent the configuration with an ordered triple (D_1, D_2, z). Two configurations (D_1, D_2, z) and $(\tilde{D}_1, \tilde{D}_2, \tilde{z})$ are said to be conformally equivalent (denoted $(D_1, D_2, z) \equiv (\tilde{D}_1, \tilde{D}_2, \tilde{z})$) if there exists a holomorphic bijection between D_2 and $\tilde{D}_2$ which further carries D_1 to $\tilde{D}_1$ and z to $\tilde{z}$.

A function $\Phi(D_1, D_2, z)$ from triples into the complex numbers is called a 'conformal invariant' if $\Phi(D_1, D_2, z) = \Phi(\tilde{D}_1, \tilde{D}_2, \tilde{z})$ whenever $(D_1, D_2, z) \equiv (\tilde{D}_1, \tilde{D}_2, \tilde{z})$. A 'conformally invariant differential of order n' is an association of a differential $\Phi(D_1, D_2, z)dz^n$ to each pair of domains as described above, which, for a holomorphic bijection $h : D_2 \longrightarrow \tilde{D}_2$, satisfies the transformation rule

$$\Phi(h(D_1), h(D_2), w)dw^n = \Phi(D_1, D_2, z)dz^n,$$

where $w = h(z)$. In other words

$$\Phi(h(D_1), h(D_2), h(z))\, h'(z)^n = \Phi(D_1, D_2, z). \tag{11}$$

One can of course also consider differentials of the form $g(z)dz^n \odot d\bar{z}^m$, or invariants that depend on a pair of points z and w (such as Green's function or the quantities appearing in Theorem 1).

Some examples follow, which can be shown to be invariants by differentiating the transformation rule $\tilde{\lambda}(h(z))|h'(z)| = \lambda(z)$ for conformal metrics under a biholomorphic change of parameter h.

Example 1. *A simple example of a conformal invariant involving the hyperbolic metrics of D_1 and D_2 is*

$$\Psi_0(D_1, D_2, z) = \frac{\lambda_2(z)}{\lambda_1(z)}.$$

In the next few examples, let Γ_1 and Γ_2 denote the Christoffel symbols of the hyperbolic metrics λ_1 and λ_2 respectively.

Example 2. *An invariant differential of order one:*

$$\Psi_1(D_1, D_2, z) = \Gamma_2(z) - \Gamma_1(z).$$

To see this, note that by (2),

$$\Gamma_i(z) = \tilde{\Gamma}_i \circ h(z)\, h'(z) + \frac{h''(z)}{h'(z)}$$

so

$$\Gamma_2(z) - \Gamma_1(z) = \left(\tilde{\Gamma}_2 \circ h(z) - \tilde{\Gamma}_1 \circ h(z)\right) h'(z).$$

Example 3. *This example is a special case of the Osgood-Stowe Schwarzian tensor* [13], [14]*:*

$$\Psi_2(D_1, D_2, z) = \frac{\partial}{\partial z}\left(\Gamma_2(z) - \Gamma_1(z)\right) + \frac{1}{2}\Gamma_1(z)^2 - \frac{1}{2}\Gamma_2(z)^2.$$

Example 4. *One can generate an infinite series of invariants using the connection ∇ corresponding to the hyperbolic metric on D_2:*

$$\Psi_{n+1}(D_1, D_2, z) = \nabla \Psi_n(D_1, D_2, z).$$

(The fact that ∇ preserves conformal invariance will be proved in Proposition 3.)

Example 5. *A familiar example of an invariant is the curvature*

$$-\lambda_1(z)^{-2} \frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda_1(z).$$

Note that this depends only on a single metric.

This previous example deserves some comment: it only depends on one domain. However, there are no non-trivial invariant differentials of the form $\Phi(D_1, z)dz^n$ depending on only one domain. It's not hard to check this by differentiating the transformation rule for the metric a few times. On the other hand, there are differentials of the form $\Phi(D_1, z)dz^n \odot d\bar{z}^m$ for m and n both positive associated to a single domain, as the above example shows. The author has verified that for the first few orders of differentiation, all the invariants are algebraic combinations of covariant derivatives of the curvature tensor, but has been unable to give a simple proof of this.

Further examples can be derived from Green's function, such as $g_1(z, w)$, $(\partial g_1/\partial z)dz$, $K_1(z, w)$, etc.

Remark 4. The conformal invariants can be thought of as moduli for the configuration described above. It's natural to ask how many moduli are required to characterize the equivalence class completely. This is easily answered in the case that D_1 and D_2 are simply connected. One can always take D_2 to be the unit disc by conformal invariance. There are as many inequivalent configurations as there are maps from the disc into itself (the image being D_1) with 0 mapping to z, modulo disc automorphisms preserving z. Thus an infinite series is required to specify the configuration up to conformal equivalence.

It is possible to show that the series given by Examples 1 through 4 characterizes the configuration up to conformal equivalence, at least in the simply connected case. We will not prove this here but just remark that it can be shown by writing the series in terms of the Taylor series of a mapping function from D_2 to D_1 as described above. In [16], universal estimates on these quantities are derived up to order seven (although the method in principle works for all odd orders.) These estimates are equivalent to distortion theorems for bounded univalent functions. (See item 2 in Section 5.)

The connections ∇ and $\overline{\nabla}$ preserve conformal invariance.

Proposition 3. *Let $\Phi(D_1, D_2, z)dz^n$ be a conformally invariant differential of order n associated with configurations (D_1, D_2, z) as described above. Define a differential of order $n+1$ as follows. Given a triple (D_1, D_2, z), let ∇ be the connection associated with the hyperbolic metric $\lambda_2(z)|dz|$ on D_2. Then, $\nabla\Phi(D_1, D_2, z)dz^n$ is also a conformally invariant differential. Similarly, if $\Phi d\bar{z}^n$ is conformally invariant, then so is $\overline{\nabla}\Phi d\bar{z}^n$.*

Proof. Given a conformal map $h : D_2 \longrightarrow \tilde{D}_2$, taking D_1 to $\tilde{D}_1$, the hyperbolic metrics on D_2 and $\tilde{D}_2$ are related via the transformation rule

$$\tilde{\lambda}_2(h(z))|h'(z)||dz| = \lambda_2(z).$$

Differentiating this equality leads to

$$\tilde{\Gamma}_2(h(z))h'(z) + \frac{h''(z)}{h'(z)} = \Gamma_2(z),$$

where $\tilde{\Gamma}_2$ and Γ_2 are the Christoffel symbols for $\tilde{\lambda}_2$ and λ respectively. On the other hand, differentiating (11) we have

$$\frac{\partial}{\partial z}\Phi(\tilde{D}_1, \tilde{D}_2, h(z))h'(z)^{n+1} + \Phi(\tilde{D}_1, \tilde{D}_2, h(z))\frac{h''(z)}{h'(z)}h'(z)^n = \frac{\partial}{\partial z}\Phi(D_1, D_2, z).$$

Denoting by $\tilde{\nabla}$ the connection corresponding to $\tilde{\lambda}_2$ and combining the two previous equalities with the definition of ∇ shows that

$$\tilde{\nabla}\Phi(\tilde{D}_1, \tilde{D}_2, w)dw^n = \nabla\Phi(D_1, D_2, z)dz^n$$

□

Remark 5. In the case that D_2 is the plane or the sphere, every conformal map of D_2 must be an affine map or Möbius transformation respectively; if D_2 is the sphere, we restrict attention to Möbius transformations which preserve the chosen association of antipodal points (i.e. isometries of λ_k).

In these cases, differentiation with respect to the corresponding connection ∇ preserves invariance under isometries of λ_k of the outer domain. This can be shown in a similar way to Proposition 3.

4. The Main inequalities

We now derive inequalities for domain functions constructed using ∇ and $\overline{\nabla}$. The summands in the expressions below are conformally invariant in the sense of Section 3 in the hyperbolic case, and in the sense of Remark 5 in the Euclidean or elliptic case.

Theorem 1. *Let D_1 and D_2 be smoothly bounded domains, $D_1 \subset D_2$, with Green's functions g_1 and g_2 respectively, and kernel functions K_i and L_i. Also, let $\zeta_\mu \in D_1$, $\mu = 1, \ldots, n$, and $\alpha_\mu \in \mathbb{C}$. Let ∇ be the connection corresponding to the outer* *domain D_2. Denote differentiation in the first variable by ∇_ζ and $\overline{\nabla}_\zeta$, and in the second by ∇_η and $\overline{\nabla}_\eta$. Then we have the following inequality:*

$$\begin{aligned}\Re\left[\sum_{\mu,\nu}\alpha_\mu\alpha_\nu(\nabla_\zeta)^m(\nabla_\eta)^m\left(L_1(\zeta_\mu,\zeta_\nu)-L_2(\zeta_\mu,\zeta_\nu)\right)\right]&\\ +\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu(\nabla_\zeta)^m(\overline{\nabla}_\eta)^m\left(K_1(\zeta_\mu,\zeta_\nu)-K_2(\zeta_\mu,\zeta_\nu)\right) &\geq 0\end{aligned}$$

This inequality also holds in the case that $D_2=\mathbb{C}$, if one replaces ∇ by ordinary differentiation and sets $L_2=L_0$ and $K_2=K_0$. If one further assumes that D_1 is elliptically k-schlicht, the inequality above holds in the case that $D_2=\overline{\mathbb{C}}$, again replacing ∇ by the connection corresponding to λ_k, replacing K_2 with K_k and L_2 by L_k.

Remark 6. The case $m=0$, for D_2 equivalent to the disc is due to Nehari [12]. Bergman and Schiffer [2] also show that the quantity

$$\Re\left[\sum_{\mu,\nu}\alpha_\mu\alpha_\nu L_1(\zeta_\mu,\zeta_\nu)\right]+\sum_{\mu,\nu}\alpha_\mu\overline{\alpha}_\nu K_1(\zeta_\mu,\zeta_\nu)$$

decreases as the domain undergoes an outward normal variation. They also prove the case that $m=0$ and the outside domain is the plane using a simple method involving positive integrals.

Much like the Grunsky inequalities, one can derive inequalities and distortion theorems for mapping functions from the above inequality. The advantage of considering higher m is that this results naturally in distortion theorems of higher order of differentiation for the mapping function.

The quantities on the left-hand side are sums over invariant differentials. To see this, recall the invariance of Green's function under a conformal map $h: D\longrightarrow\tilde{D}$ of domains

$$\tilde{g}(h(z),h(w))=g(z,w).$$

Differentiating this shows that $L_i(z,w)dz\odot dw$ and $K_i(z,w)dz\odot d\bar{w}$ are invariant differentials. One then applies Proposition 3 or Remark 5.

Proof. (of Theorem 1.) We first prove the case that D_2 is not the plane or the sphere. Let

$$p_i(\zeta)=\Re\left(\sum_\nu\alpha_\nu\,(\nabla_\eta)^m|_{\eta=\zeta_\nu}\,g_i(\zeta,\eta)\right).$$

Consider the piecewise continuous function

$$\epsilon(\zeta)=\begin{cases}p_2(\zeta)-p_1(\zeta) & : \ \zeta\in D_1\\ p_2(\zeta) & : \ \zeta\in D_2\backslash D_1\end{cases}$$

The positivity of the Dirichlet energy of this quantity is the source of the inequality of the theorem. We now compute this energy using Green's theorem. Let n denote the outward unit normal (in ζ).

$$\begin{aligned}\iint_{\partial D_2}\nabla\epsilon\cdot\nabla\epsilon\,dA_\zeta &= \int_{\partial D_2}p_2\frac{\partial p_2}{\partial n}\,ds-\int_{\partial D_1}p_2\frac{\partial p_2}{\partial n}\,ds\\ &+\int_{\partial D_1}(p_2-p_1)\left(\frac{\partial p_2}{\partial n}-\frac{\partial p_1}{\partial n}\right)ds\end{aligned}$$

Using the fact that $p_i|_{\partial D_i} = 0$, and the positivity of the Dirichlet energy, we get that

$$0 \leq \int_{\partial D_1} (p_1 - p_2) \frac{\partial p_1}{\partial n}\, ds \tag{12}$$

We rewrite this integrand in order to make use of Proposition 2. First, note that since $\overline{\nabla_\eta f(z)} = \overline{\nabla}_\eta \overline{f(z)}$ for any complex valued function $f(z)$,

$$p_1 = \frac{1}{2} \sum_\nu \Big(\alpha_\nu \, (\nabla_\eta)^m\big|_{\eta=\zeta_\nu} g_1(\zeta,\eta) + \overline{\alpha}_\nu \, (\overline{\nabla}_\eta)^m\big|_{\eta=\zeta_\nu} g_1(\zeta,\eta) \Big).$$

Since $p_1 = 0$ on ∂D_1,

$$\frac{\partial p_1}{\partial n}\, ds = \frac{2}{i} \frac{\partial p_1}{\partial \zeta}\, ds,$$

and so

$$\frac{\partial p_1}{\partial n_\zeta}\, ds_\zeta = \frac{1}{2} \sum_\nu \left(\alpha_\nu \, (\nabla_\eta)^m\big|_{\eta=\zeta_\nu} \frac{\partial g_1}{\partial \zeta}(\zeta,\eta) + \overline{\alpha}_\nu \, (\overline{\nabla}_\eta)^m\big|_{\eta=\zeta_\nu} \frac{\partial g_1}{\partial \zeta}(\zeta,\eta) \right).$$

So

$$\begin{aligned} \int_{\partial D_1} (p_1 - p_2) \frac{\partial p_1}{\partial n_\zeta}\, ds_\zeta = \Re \Bigg(\frac{1}{i} \int_{\partial D_1} & \sum_\nu \alpha_\nu \, (\nabla_\eta)^m\big|_{\eta=\zeta_\nu} (g_1(\zeta,\eta) - g_2(\zeta,\eta)) \\ & \cdot \left(\sum_\nu \alpha_\nu \, (\nabla_\eta)^m\big|_{\eta=\zeta_\nu} \frac{\partial g_1}{\partial \zeta}(\zeta,\eta) + \overline{\alpha}_\nu \, (\overline{\nabla}_\eta)^m\big|_{\eta=\zeta_\nu} \frac{\partial g_1}{\partial \zeta}(\zeta,\eta) \right) d\zeta \Bigg) \end{aligned}$$

Now $(g_1 - g_2)$ is harmonic on D_1 and extends continuously to the boundary since the boundary is smooth. Thus by Proposition 2,

$$\begin{aligned} \Re \Bigg(\sum_{\mu,\nu} \alpha_\mu \alpha_\nu (\nabla_\zeta)^m (\nabla_\eta)^m (g_2 - g_1)(\zeta_\mu, \zeta_\nu) \Bigg) & \\ + \sum_{\mu,\nu} \alpha_\mu \overline{\alpha}_\nu (\nabla_\zeta)^m (\overline{\nabla}_\eta)^m (g_2 - g_1)(\zeta_\mu, \zeta_\nu) & \quad \geq \quad 0 \end{aligned}$$

which finishes the proof in the special case that D_2 is not the plane or the sphere.

In the case that D_2 is the sphere, one constructs ϵ slightly differently. Let D_1^* denote the domain consisting of points antipodal to D_1, and p_1^* be given by

$$p_1^*(\zeta) = -p_1(-(k\bar{\zeta})^{-1}),$$

and finally let

$$p_2(\zeta) = \Re \Bigg(\sum_\nu \alpha_\nu \, (\nabla_\eta)^m\big|_{\eta=\zeta_\nu} g_k(\zeta,\eta) \Bigg).$$

Define ϵ by

$$\epsilon(\zeta) = \begin{cases} p_1(\zeta) - p_2(\zeta) & : \quad \zeta \in D_1 \\ -p_2(\zeta) & : \quad \zeta \in \overline{\mathbb{C}} \setminus (D_1 \cup D_1^*) \\ p_1^*(\zeta) - p_2(\zeta) & : \quad \zeta \in D_1^* \end{cases}$$

Doing a similar computation to that above and using the fact that $p_2(-(k\bar{\zeta})^{-1})) = -p_2(\zeta)$, one arrives at

$$0 \leq \int_{\partial D_1} (p_1 - p_2) \frac{\partial p_1}{\partial n}\, ds.$$

Replacing g_2 with g_k, the rest of the proof following (12) is identical.

Finally, in the case of the plane, D_1 can be bounded in a disc of sufficiently high radius $1/\sqrt{|k|}$; since for $k < 0$, g_k is the actual Green's function of this disc, we can apply the first case and let $k \to \infty$. □

Remark 7. It is possible to weaken the assumption that D_1 is smoothly bounded but we will not pursue this here.

5. Discussion of corollaries and previous results

(1) Theorem 1 holds if one replaces ∇ and $\overline{\nabla}$ with $\partial/\partial z$ and $\partial/\partial \bar{z}$ respectively. This was proved in [16] in the case that both domains are equivalent to the disc, and in [17] in the case that the outer domain is $\mathbb{C}$ or $\overline{\mathbb{C}}$. When one makes this replacement, the quantities in the inequalities are no longer invariant in the sense of Section 3 or Remark 5 in the hyperbolic case or the Euclidean and elliptic cases respectively. On the other hand, the quantities are holomorphic or anti-holomorphic in each variable.

(2) By choosing values of α_μ and ζ_μ, and writing the quantities in Theorem 1 in terms of the mapping function, one can derive estimates on the corresponding class of mappings. For a well-known example, consider the case that D_2 is the unit disc, and f is a univalent map from the disc onto D_1. Set $m = 0$, $n = 1$, $\alpha = e^{i\theta}$, and $\zeta = f(z)$; this results in the inequality

$$\left| \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \right| \leq 6\left(1 - \frac{|f'(z)|^2(1-|z|^2)^2}{(1-|f(z)|^2)^2}\right).$$

For $m > 0$ one gets higher-order distortion theorems for bounded univalent functions; roughly, one gets distortion theorems whose order of differentiation is $2m + 3$. In [16] sharp distortion theorems of this kind were derived from a set of inequalities similar to Theorem 1 (described in the previous item).

The distortion theorems in [16], when written in terms of the invariants in Example 4, can be interpreted as higher-order Schwarz lemmas. In particular, one can estimate the change in geodesic curvature of a curve, and its derivatives, under a bounded univalent map. The idea of higher-order Schwarz lemmas seems to be due to Flinn and Osgood [3].

(3) A natural question is, are there inequalities such as Theorem 1 with an odd order of differentiation? Such odd-order inequalities were proven in [18] using a modification of Hadamard variation. The quantities involved in these inequalities are unfortunately not all conformally invariant; on the other hand, they are holomorphic or anti-holomorphic in both variables.

6. Open questions

(1) The Ahlfors-Beurling theorem provides a link between extremal length and Dirichlet energy. So a natural question is: can one give extremal metric interpretations of the conformal invariants in Example 4 or Theorem 1? (This question is one of the reasons for the insistence on conformal invariance.)

One can interpret Example 1 in terms of the reduced modules of D_1 and D_2. The monotonicity of reduced module is a form of the Schwarz lemma. Can one give higher-order extremal metric quantities that produce 'higher-order' geometric distortion theorems?

It may not be necessary to restrict to length-area methods; Ahlfors [1] and Minda [10] have developed a 'geodesic curvature-area' method for computing modules. There is a connection between the quantities in Examples 2 and 3 and curvature [16], so it's not a stretch that modifications of these geodesic curvature-area methods might produce such invariants.

(2) Can one prove inequalities such as those in Theorem 1 with odd order of differentiation, that are also conformally invariant?

(3) Can one identify the connectedness of, say, the inner domain from the conformal invariants? For example, setting $D_2 = \mathbb{C}$ and treating $L_1 - L_2$ as an bounded operator on the Bergman space, it was shown by Bergman and Schiffer [2] that the dimension of the 1-eigenspace of $L_1 - L_2$ is $n - 1$ if the connectivity of the inner domain is n. (This is closely related to the fact that there are $n - 1$ linearly independent harmonic functions with constant boundary values on an n-connected domain.) Do higher-order invariants contain information about connectivity?

References

[1] Ahlfors, L. V. *Geodesic curvature and area.* (1962), Studies in mathematical analysis and related topics 1–7 Stanford Univ. Press, Stanford, Calif.

[2] Bergman, S. and Schiffer, M. *Kernel functions and conformal mapping.* Compositio Math. **8** (1951), 205–249.

[3] Flinn, B. and Osgood, B. *Hyperbolic curvature and conformal mapping.* Bull. London Math. Soc. **18** (1986), no. 3, 272–276.

[4] Duren, P. and Kühnau, R. *Elliptic capacity and its distortion under conformal mapping.* J. Anal. Math. **89** (2003), 317–335.

[5] Grunsky, H. *Einige Analoga zum Schwarzschen Lemma.* Math. Ann. **108** (1933), 190–196.

[6] Kühnau, R. *Über vier Klassen schlicter Funktionen.* Math. Nachr. **50** (1971), 17–26.

[7] Ma, W. and Minda, D. *Euclidean linear invariance and uniform local convexity.* J. Austral. Math. Soc. (Series A) **52** (1992), 401–418.

[8] Ma, W. and Minda, D. *Hyperbolic linear invariance and hyperbolic k-convexity.* J. Austral. Math. Soc. (Series A) **58** (1995), 73–93.

[9] Minda, D. *unpublished notes.*

[10] Minda, D. *The modulus of a doubly connected region and the geodesic curvature-area method.* Pacific J. Math. **113** (1984), no. 2, 395–406.

[11] Nakahara, M. *Geometry, topology and physics.* Graduate Student Series in Physics. Adam Hilger, Ltd., Bristol, 1990.

[12] Nehari, Z. *Some inequalities in the theory of functions.* Trans. Amer. Math. Soc. **75** (1953), 256–286.

[13] Osgood, B. and Stowe, D. *A generalization of Nehari's univalence criterion.* Comment. Math. Helv. **65** (1990), no. 2, 234–242.

[14] Osgood, B. and Stowe, D. *The Schwarzian derivative and conformal mapping of Riemannian manifolds.* Duke Math. J. **67** (1992), no. 1, 57–99

[15] Peschl, Ernst. *Les invariants différentiels non holomorphes et leur role dans la théorie des fonctions.* Rend. Sem. Mat. Messina **1** (1955), 100–108.

[16] Schippers, E. *Conformal invariants and higher-order Schwarz lemmas.* J. Anal. Math. **90** (2003), 217–242

[17] Schippers, E. *Estimates on kernel functions of elliptically schlicht domains.* preprint.

[18] Schippers, E. *Hadamard variation and inequalities for multiply-connected domains.* preprint.