Derivative of the Nehari functional

Eric Schippers

Department of Mathematics
University of Manitoba
Winnipeg, Canada

Baylor University 2009

A wide-angle view

Describe the boundary of the class of normalized univalent functions $f : \mathbb{D} \to \mathbb{D}$ or $\mathbb{C}$.

boundary points $\leftrightarrow$ functionals $\leftrightarrow$ quadratic differentials

A wide-angle view

Describe the boundary of the class of normalized univalent functions $f : \mathbb{D} \to \mathbb{D}$ or $\mathbb{C}$.

boundary points $\leftrightarrow$ functionals $\leftrightarrow$ quadratic differentials

Teichmüller's principle says that a functional involving derivatives of order $n+1$ at a point “is associated with” a quadratic differential with a pole of order n.

A wide-angle view

Describe the boundary of the class of normalized univalent functions $f : \mathbb{D} \to \mathbb{D}$ or $\mathbb{C}$.

boundary points $\leftrightarrow$ functionals $\leftrightarrow$ quadratic differentials

Teichmüller's principle says that a functional involving derivatives of order $n+1$ at a point "is associated with" a quadratic differential with a pole of order n.

Schiffer's variational method shows that the derivative of a functional of order $n+1$ involves a quadratic differential of order n.

Jenkins'/Teichmüller's extremal metric method *produces* a functional, given a quadratic differential.

Why hope for anything more specific?

Case $\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ one-to-one},\ f(0) = f'(0) - 1 = 0\}$.

Pfluger showed that for the third coefficient body

$$\{(a_2^2, a_3) :\ \exists f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}\}$$

the association between boundary points, third-order homogeneous functionals, and quadratic differentials is one-to-one (up to a rotation).

Purpose of this talk

- Given a singular test function, **Nehari's method** produces a monotonic, bounded functional on the class of bounded univalent functions $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$.

Purpose of this talk

- Given a singular test function, **Nehari's method** produces a monotonic, bounded functional on the class of bounded univalent functions $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$.
- It can be shown that Nehari's method produces a functional, given a quadratic differential (like the extremal metric method).

Purpose of this talk

- Given a singular test function, **Nehari's method** produces a monotonic, bounded functional on the class of bounded univalent functions $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$.
- It can be shown that Nehari's method produces a functional, given a quadratic differential (like the extremal metric method).
- Schiffer's insight was: the *derivative* of a functional is associated with a quadratic differential.

Purpose of this talk

- Given a singular test function, **Nehari's method** produces a monotonic, bounded functional on the class of bounded univalent functions $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$.
- It can be shown that Nehari's method produces a functional, given a quadratic differential (like the extremal metric method).
- Schiffer's insight was: the *derivative* of a functional is associated with a quadratic differential.
- **Question**: what is the derivative of Nehari's functional? Is it the original quadratic differential?

Purpose of this talk

- Given a singular test function, **Nehari's method** produces a monotonic, bounded functional on the class of bounded univalent functions $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$.
- It can be shown that Nehari's method produces a functional, given a quadratic differential (like the extremal metric method).
- Schiffer's insight was: the *derivative* of a functional is associated with a quadratic differential.
- **Question**: what is the derivative of Nehari's functional? Is it the original quadratic differential?

I'll answer this question.

Definition of Nehari's functional

Extended truncated Dirichlet inner product:

$$(g,h) = \sum_{k=-n}^{n} k\overline{g}_k h_k, \text{ for } g(z) = \sum_{k=-n}^{\infty} g_k z^k, \; h(z) = \sum_{k=-n}^{\infty} h_k z^k.$$

Definition (Nehari's functional)

- Let $x(z) = \sum_{k=-n}^{n} x_k z^k$ be a test function on $\mathbb{D}$, satisfying $\mathrm{Re}(x'(z)dz) = 0$ on $\partial\mathbb{D}$.
- Let $f : \mathbb{D} \to \mathbb{D}$ be one-to-one, and satisfy $f(0) = 0$.
- Let $y = x \circ f$.

$$\mathrm{Neh}(f) = (x,x) - (y,y) + \left\| \sum_{k=1}^{n} (y_k + \overline{y_{-k}}) z^k \right\|_{\mathbb{D}}^2 .$$

Some assumptions

Nehari's method is more general than above.

Assumptions:

- The test function has a pole only at 0.
- The outside domain is the disc, and the inside domain is simply connected.

These choices lead to coefficient estimates for bounded univalent functions.

Some assumptions

Nehari's method is more general than above.

Assumptions:

- The test function has a pole only at 0.
- The outside domain is the disc, and the inside domain is simply connected.

These choices lead to coefficient estimates for bounded univalent functions.

Crucial point: The condition $\mathrm{Re}(x'(z)dz) = 0$ on $\partial\mathbb{D}$ holds $\Leftrightarrow x_k = -\overline{x_{-k}} \Leftrightarrow x'(z)^2 dz^2$ is a quadratic differential on the disc.

Some assumptions

Nehari's method is more general than above.
Assumptions:

- The test function has a pole only at 0.
- The outside domain is the disc, and the inside domain is simply connected.

These choices lead to coefficient estimates for bounded univalent functions.

Crucial point: The condition $\mathrm{Re}(x'(z)dz) = 0$ on $\partial\mathbb{D}$ holds $\Leftrightarrow x_k = -\overline{x_{-k}} \Leftrightarrow x'(z)^2dz^2$ is a quadratic differential on the disc.

This can *always* be arranged by subtracting the lower bound of the functional.

The case of equality

$$\begin{aligned} \mathrm{Neh}(f) &= (x,x) - (y,y) + \left\| \sum_{k=1}^{n} (y_k + \bar{y}_{-k}) z^k \right\|_{\mathbb{D}}^2 \\ &= \|x\|^2_{\mathbb{D}\setminus f(\mathbb{D})} + \left\| \sum_{k=n+1}^{\infty} y_l z^l \right\|_{\mathbb{D}}^2 + \left\| \sum_{k=1}^{n} (y_k + \overline{y_{-k}}) z^k \right\|_{\mathbb{D}}^2 \end{aligned}$$

The case of equality

$$\begin{aligned} \mathrm{Neh}(f) &= (x,x) - (y,y) + \left\| \sum_{k=1}^{n} (y_k + \bar{y}_{-k}) z^k \right\|_{\mathbb{D}}^2 \\ &= \|x\|^2_{\mathbb{D}\setminus f(\mathbb{D})} + \left\| \sum_{k=n+1}^{\infty} y_l z^l \right\|_{\mathbb{D}}^2 + \left\| \sum_{k=1}^{n} (y_k + \overline{y_{-k}}) z^k \right\|_{\mathbb{D}}^2 \end{aligned}$$

- Inequality $\mathrm{Neh}(f) \geq 0$ is *weaker* than the area principal
- But equality is more difficult to attain.

The case of equality

$$\begin{aligned}\mathrm{Neh}(f) &= (x,x) - (y,y) + \left\| \sum_{k=1}^{n} (y_k + \bar{y}_{-k}) z^k \right\|_{\mathbb{D}}^2 \\ &= \|x\|^2_{\mathbb{D}\setminus f(\mathbb{D})} + \left\| \sum_{k=n+1}^{\infty} y_l z^l \right\|_{\mathbb{D}}^2 + \left\| \sum_{k=1}^{n} (y_k + \overline{y_{-k}}) z^k \right\|_{\mathbb{D}}^2\end{aligned}$$

- Inequality $\mathrm{Neh}(f) \geq 0$ is *weaker* than the area principal
- But equality is more difficult to attain.
- Equality holds if and only if f is admissible for the quadratic differential $x'(z)^2 dz^2$

The case of equality

$$\begin{aligned}\text{Neh}(f) &= (x,x) - (y,y) + \left\| \sum_{k=1}^{n} (y_k + \bar{y}_{-k}) z^k \right\|_{\mathbb{D}}^2 \\ &= \|x\|^2_{\mathbb{D}\setminus f(\mathbb{D})} + \left\| \sum_{k=n+1}^{\infty} y_l z^l \right\|_{\mathbb{D}}^2 + \left\| \sum_{k=1}^{n} (y_k + \overline{y_{-k}}) z^k \right\|_{\mathbb{D}}^2\end{aligned}$$

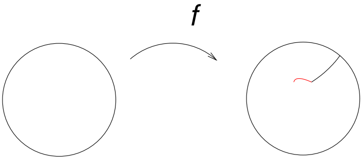
- Inequality $\text{Neh}(f) \geq 0$ is *weaker* than the area principal
- But equality is more difficult to attain.
- Equality holds if and only if f is admissible for the quadratic differential $x'(z)^2 dz^2$
  Proof: f admissible for $x'(z)^2 dz^2 \Leftrightarrow (x \circ f)'(z)^2 dz^2$ is a quadratic differential on the disc $\Leftrightarrow y_k = -\overline{y_{-k}}$ for all k

Monotonicity and arc truncation

Theorem (Nehari)

$$f_1(\mathbb{D}) \subset f_2(\mathbb{D}) \Rightarrow Neh(f_1) \geq Neh(f_2).$$

Thus equality must continue to hold under arc truncation.

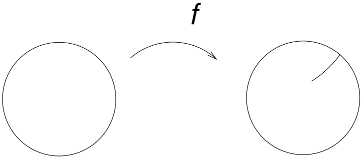


Monotonicity and arc truncation

Theorem (Nehari)

$$f_1(\mathbb{D}) \subset f_2(\mathbb{D}) \Rightarrow Neh(f_1) \geq Neh(f_2).$$

Thus equality must continue to hold under arc truncation.



The power matrix

Let $f(z)$ be univalent in a neighbourhood of 0 and $f(0) = 0$.

Definition (Power matrix)

The power matrix of f is the matrix $[f]$ defined by
$f(z)^m = [f]^m_m z^m + [f]^m_{m+1} z^{m+1} + \cdots$.

Great thing about the power matrix: $[f \circ g] = [f][g]$.

The power matrix

Let $f(z)$ be univalent in a neighbourhood of 0 and $f(0) = 0$.

Definition (Power matrix)

The power matrix of f is the matrix $[f]$ defined by
$f(z)^m = [f]_m^m z^m + [f]_{m+1}^m z^{m+1} + \cdots$.

Great thing about the power matrix: $[f \circ g] = [f][g]$.

Used by

- Jabotinsky
- Schiffer and Tammi
- Friedland and Schiffer

It simplifies many function theoretic computations.

Infinitesimal power matrices (Lie algebra)

Let $h(z) = h_1 z + h_2 z^2 + \cdots$.

Definition

The infinitesimal power matrix of h is

$$\langle h \rangle = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & -h_1 & -h_2 & -h_3 & -h_4 & -h_5 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & h_1 & h_2 & h_3 & \cdots \\ \cdots & 0 & 0 & 0 & 2h_1 & 2h_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 3h_1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Friedland-Schiffer equation

Let $\mathrm{Re}\, p_t(z) > 0$, $p_t(0) = 1$, $t \geq 0$.
The solution of the **Friedland-Schiffer differential equation**

$$\frac{\partial f_t}{\partial t}(z) = -zp(z)\frac{\partial f_t}{\partial z}(z) \quad f_0(z) = z$$

satisfies $f : \mathbb{D} \to \mathbb{D}$ and $f_t(\mathbb{D}) \subset f_s(\mathbb{D})$ whenever $t > s$.

Friedland-Schiffer equation

Let $\text{Re} p_t(z) > 0$, $p_t(0) = 1$, $t \geq 0$.
The solution of the **Friedland-Schiffer differential equation**

$$\frac{\partial f_t}{\partial t}(z) = -zp(z)\frac{\partial f_t}{\partial z}(z) \quad f_0(z) = z$$

satisfies $f : \mathbb{D} \to \mathbb{D}$ and $f_t(\mathbb{D}) \subset f_s(\mathbb{D})$ whenever $t > s$.

In the power matrix notation, the coefficients of f_t satisfy:

$$\frac{d}{dt}[f_t] = -[f_t]\left\langle zp_t \right\rangle .$$

An example computation

For a test function $x(z)$ denote $\mathbf{x} = (x_{-n}, \ldots, x_n)$ and $y_t = x \circ f_t$. Then (suitably truncating the power matrices)

$$\begin{aligned} \frac{d}{dt}(\mathbf{y}, \mathbf{y}) &= \frac{d}{dt}(\mathbf{x}[f_t], \mathbf{x}[f_t]) \\ &= -2\mathrm{Re}\,(\mathbf{x}[f_t], \mathbf{x}[f_t]\,\langle zp_t \rangle) \\ &= -2\mathrm{Re}\,(\mathbf{y}, \mathbf{y}\,\langle zp_t \rangle) \end{aligned}$$

Note: all vectors are row vectors and operators appear on their right.

Differentiating Nehari's functional

Nehari's functional in power matrix notation.
Reflection: $(y_{-n}, \ldots, y_n)R = (\overline{y_n}, \ldots, \overline{y_{-n}})$
Projection: $(y_{-n}, \ldots, y_n)P = (y_{-n}, \ldots, y_{-1}, 0, \ldots, 0)$.

With this notation, for $f : \mathbb{D} \to \mathbb{D}$, f one-to-one, $f(0) = 0$, $y = x \circ f$

$$\mathrm{Neh}(f) = (\mathbf{x}, \mathbf{x}) - \mathrm{Re}\,(\mathbf{y}(I + R), \mathbf{y}P)\,.$$

Differentiating Nehari's functional

Nehari's functional in power matrix notation.
Reflection: $(y_{-n},\ldots,y_n)R=(\overline{y_n},\ldots,\overline{y_{-n}})$
Projection: $(y_{-n},\ldots,y_n)P=(y_{-n},\ldots,y_{-1},0,\ldots,0)$.
With this notation, for $f:\mathbb{D}\to\mathbb{D}$, f one-to-one, $f(0)=0$, $y=x\circ f$

$$\mathrm{Neh}(f)=(\mathbf{x},\mathbf{x})-\mathrm{Re}\,(\mathbf{y}(I+R),\mathbf{y}P)\,.$$

Theorem (Functional derivative of Neh, S 2009)

Let f_t be a solution of the Friedland-Schiffer equation with infinitesimal generator p_t. Then

$$\frac{d}{dt}Neh(f_t)=-Re\,(\mathbf{y}_t(I+R),\mathbf{y}_t\,[P,\langle zp_t\rangle])+Re\,(\mathbf{y}_t,\mathbf{y}_t\,(\langle zp_t\rangle+\langle zp_t\rangle^*)\,P)\,.$$

Derivative at an extremal

At an extremal f_s, $y_s'(z)^2 dz^2$ is a quadratic differential for which $\partial\mathbb{D}$ is a trajectory. So $y_k = -\bar{y}_{-k}$ so $\mathbf{y}_s R = -\mathbf{y}_s$. So

$$\begin{aligned} \frac{d}{dt}\mathrm{Neh}(f_t)\Big|_{t=s} &= -\mathrm{Re}\left(\mathbf{y}_s(I+R), \mathbf{y}_s\left[P, \langle zp_s\rangle\right]\right) \\ &\quad +\mathrm{Re}\left(\mathbf{y}_s, \mathbf{y}_s\left(\langle zp_s\rangle + \langle zp_s\rangle^*\right)P\right) \\ &= \mathrm{Re}\left(\mathbf{y}_s, \mathbf{y}_s\left(\langle zp_s\rangle + \langle zp_s\rangle^*\right)P\right) \\ &= 2\mathrm{Re}\left(\mathbf{y}_s, \mathbf{y}_s\langle zp_s\rangle\right). \end{aligned}$$

Note: This is true whether or not f_t continues to be extremal for $t > s$. i.e. This is the derivative in *any* direction p_s.

Where's the quadratic differential?

In this case,

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=s} \mathrm{Neh}(f_t) &= 2\mathrm{Re}\left(\mathbf{y}_s, \mathbf{y}_s \langle z p_s \rangle\right) \\
&= 2\mathrm{Re} \lim_{r \to 1^-} \frac{1}{2\pi i} \int_{\gamma_r} (z y_s')^*(z) \cdot z y_s'(z) \cdot p_s(z) \frac{dz}{z} \\
&= \lim_{r \to 1^-} \mathrm{Re} \frac{1}{\pi i} \int_{\gamma_r} z^2 y_s'(z)^2 \cdot p_s(z) \frac{dz}{z}.
\end{aligned}
$$

Where's the quadratic differential?

In this case,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=s} \mathrm{Neh}(f_t) &= 2\mathrm{Re}\left(\mathbf{y}_s, \mathbf{y}_s \langle z p_s \rangle\right) \\
&= 2\mathrm{Re} \lim_{r\to 1^-} \frac{1}{2\pi i} \int_{\gamma_r} (z y_s')^*(z) \cdot z y_s'(z) \cdot p_s(z) \frac{dz}{z} \\
&= \lim_{r\to 1^-} \mathrm{Re} \frac{1}{\pi i} \int_{\gamma_r} z^2 y_s'(z)^2 \cdot p_s(z) \frac{dz}{z}.
\end{aligned}$$

Recall that $Q_s(z)dz^2/z^2 = y_s'(z)^2 dz^2$ is a quadratic differential. So

$$\left.\frac{d}{dt}\right|_{t=r} \mathrm{Neh}(f_t) = \lim_{r\to 1^-} \mathrm{Re} \frac{1}{\pi i} \int_{\gamma_r} Q_s(z) \cdot p_s(z) \frac{dz}{z}.$$

Action of quadratic differential is natural

Quadratic differentials act naturally on infinitesimal generators:

$$\frac{Q(z)}{z^2}dz^2\left(zp(z)\frac{\partial}{\partial z}\right) \mapsto Q(z)\cdot p(z)\frac{dz}{z}$$

Now integrate:

$$\lim_{r\to 1}\int_{\gamma^r} Q(z)p(z)\frac{dz}{z}.$$

Action of quadratic differential is natural

Quadratic differentials act naturally on infinitesimal generators:

$$\frac{Q(z)}{z^2}dz^2\left(zp(z)\frac{\partial}{\partial z}\right) \mapsto Q(z)\cdot p(z)\frac{dz}{z}$$

Now integrate:

$$\lim_{r\to 1}\int_{\gamma^r} Q(z)p(z)\frac{dz}{z}.$$

Nice formula in terms of coefficients: If
$Q(z) = q_{-n}z^{-n} + q_{-n+1}z^{-n+1} + \cdots$
and $p(z) = 1 + p_1 z + \cdots$ then the integral is

$$\sum_{k=1}^{n} q_{-k}p_k.$$

Summary

- **Schiffer**: functional $\rightarrow$ quadratic differential.

Summary

- **Schiffer**: functional $\rightarrow$ quadratic differential.
- **Nehari** or **Jenkins/Teichmüller**: quadratic differential $\rightarrow$ functional.
- but a quadratic differential is supposed to be a derivative.

Summary

- **Schiffer**: functional $\to$ quadratic differential.
- **Nehari** or **Jenkins/Teichmüller**: quadratic differential $\to$ functional.
- but a quadratic differential is supposed to be a derivative.

Punch line: The derivative of the Nehari functional at an extremal is the pull-back of the original “input” quadratic differential under the extremal map.