CONVERGENCE OF THE WEIL-PETERSSON METRIC ON THE TEICHMÜLLER SPACE OF BORDERED RIEMANN SURFACES

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Abstract. Let Σ be a Riemann surface of genus g bordered by n curves homeomorphic to the circle $\mathbb{S}^1$, and assume that $2g + 2 - n > 0$. For such bordered Riemann surfaces, the authors have previously defined a Teichmüller space which is a Hilbert manifold and which is holomorphically included in the standard Teichmüller space. We show that this Teichmüller space has a convergent Weil-Petersson metric.

We also present an alternate model of the aforementioned Teichmüller space by showing that it is locally modelled on a space of L^2 Beltrami differentials, which are holomorphic up to a power of the hyperbolic metric.

1. Introduction

1.1. **Introduction and statement of results.** In this paper, we demonstrate that the refined Teichmüller space of bordered Riemann surfaces defined by the authors in [18] possesses a convergent Weil-Petersson metric, and a simple L^2 space of Beltrami differentials models the tangent space.

The Weil-Petersson metric converges on finite-dimensional Teichmüller spaces. However it has long been known to diverge on more general Teichmüller spaces. S. Nag and A. Verjovsky [13] showed that the metric does in fact converge in directions tangent to the subset of the universal Teichmüller space which correspond to analytic parametrizations of $\mathbb{S}^1$. Later, G. Cui [2] and G. Hui [7], and independently L. Takhtajan and L-P. Teo [22], found the completion of this space which is modelled on L^2 Beltrami differentials, or the space of univalent functions whose pre-Schwarzian is in the Bergman space of the unit disk. Takhtajan and Teo showed that the WP-class Teichmüller space is a Hilbert manifold locally modelled on a certain space of quadratic differentials and is a topological group. They also gave explicit Kähler potentials for the WP-metric, among many other results. Since then there has been great interest in what has come to be called the WP-class universal Teichmüller space.

In this paper we extend convergence of the Weil-Petersson metric to a Teichmüller space of Riemann surfaces with g handles and n boundary curves homeomorphic to $\mathbb{S}^1$. One might think that the aforementioned results on the universal Teichmüller space can be passed down to arbitrary quotients to obtain convergence on arbitrary Teichmüller spaces. However, this is not the case. Given a bordered Riemann surface, the relevant class of quadratic differentials are L^2 on the surface itself, and hence on the fundamental domain of the associated Fuchsian group. The lift of such a quadratic differential to the universal cover is not in L^2 unless the L^2 integral is zero on each fundamental domain (see Remark 4.4 ahead).

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To show that this Teichmüller space possesses a convergent Weil-Petersson metric, we use a new technique. We take advantage of a fiber structure discovered by two of the authors [16], which arises from an identification of the Teichmüller space of bordered surfaces with a moduli space of Friedan-Shenker-Vafa [14]. The authors used this fiber structure in [18] to construct a Hilbert manifold structure on the refined Teichmüller space of bordered surfaces of this type. In this paper we show that this Hilbert manifold is locally modelled on a space of L^2 Beltrami differentials on the base surface that are holomorphic up to a power of the hyperbolic metric. Furthermore we will express the Weil-Petersson inner product in terms of these differentials. Our arguments rely heavily on sewing techniques developed in [14] and [16].

2. Bordered Riemann surfaces and collars

2.1. **Differentials on bordered surfaces.** First we establish some notation for the various function spaces involved. Let Σ be a Riemann surface with a hyperbolic metric. Let $\{\phi_U : U \to \mathbb{C}\}$ be an atlas of local biholomorphic parameters covering Σ. For $k, l \in \mathbb{Z}$ a (k,l)-differential h is a collection of functions $h_U : \phi_U(U) \to \mathbb{C}$ such that whenever $U \cap V$ is non-empty, denoting by $z = g(w) = \phi_V \circ \phi_U^{-1}(w)$ the change of parameter, the functions h_U and h_V satisfy the transformation rule

$$h_V(w) g'(w)^k \overline{g'(w)}^l = h_U(z); \tag{2.1}$$

that is, h has the expression $h_U(z) dz^k d\bar{z}^l$ in local coordinates.

The hyperbolic metric induces a norm on (k,l)-differentials at any point in the following way. Setting $m = k + l$, at a fixed point in Σ the quantity

$$|h_U(z)|^p \rho_U(z)^{2-mp}$$

is independent of choice of coordinates z, where $\rho_U(z)^2 \, |dz|^2$ is the local expression for the hyperbolic metric. By integrating over any subset $W \subset \Sigma$ and taking the pth root we obtain a well-defined hyperbolic L^p norm $\|h\|_{p,W}$ on W.

Definition 2.1. Let $W \subset \Sigma$ be an open set. For $1 \leq p \leq \infty$, let

$$L^p_{k,l}(\Sigma, W) = \{(k,l) - \text{differentials } \ h \ : \ \|h\|_{p,W} < \infty\}.$$

Let

$$A^p_k(\Sigma, W) = \{h \in L^p_{k,0}(\Sigma, W) \ : \ h \ \text{ holomorphic}\}.$$

Denote $L^p_{k,l}(\Sigma, \Sigma)$ by $L^p_{k,l}(\Sigma)$ and $A^p_k(\Sigma, \Sigma)$ by $A^p_k(\Sigma)$.

Remark 2.2. We will not distinguish the norms $\| \cdot \|_{p,W}$ notationally with respect to the order of the differential, since the type of differential uniquely determines the norm. If the subscript "W" is omitted, it is assumed that $W = \Sigma$.

2.2. **WP-class quasisymmetries and quasiconformal maps.** In [18] the authors defined a Teichmüller space of bordered surfaces which possesses a Hilbert manifold structure. We briefly review some of the definitions and results, as well as introduce new definitions necessary in the next few sections.

Let

$$\mathbb{D} = \{z \ : \ |z| < 1\}, \quad \mathbb{D}^* = \{z \ : \ |z| > 1\} \cup \{\infty\}, \quad \text{and} \quad \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Definition 2.3. Let $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ denote the set of holomorphic one-to-one maps $f : \mathbb{D} \to \mathbb{C}$ with quasiconformal extensions to $\bar{\mathbb{C}}$ such that $(f''(z)/f'(z))dz \in A^2_1(\mathbb{D})$ and $f(0) = 0$.

By results of Takhtajan and Teo, the image of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ under the map

$$f \longmapsto \left(\frac{f''(z)}{f'(z)}, f'(0) \right) \tag{2.2}$$

is an open subset of the Hilbert space $A^2_1(\mathbb{D}) \oplus \mathbb{C}$ with the direct sum inner product [17, Theorem 2.3].

Elements of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ arise as conformal maps associated to quasisymmetries in the following way. Given a quasisymmetry $\phi : \mathbb{S}^1 \to \mathbb{S}^1$, by the Ahlfors-Beurling extension theorem, there exists a quasiconformal map $w : \mathbb{D}^* \to \mathbb{D}^*$ such that $w|_{\mathbb{S}^1} = \phi$. This quasiconformal map has complex dilatation

$$\mu = \frac{\overline{\partial} w}{\partial w} \in L^\infty_{-1,1}(\mathbb{D}^*).$$

Let f^μ be the solution to the Beltrami equation

$$\frac{\overline{\partial} f}{\partial f} = \hat{\mu}$$

where $\hat{\mu}$ is the Beltrami differential which equals μ on $\mathbb{D}^*$ and 0 on $\mathbb{D}$. We normalize f^μ so that $f^\mu(0) = 0$, $f^\mu(\infty) = \infty$ and $f^{\mu\prime}(\infty) = 1$ for definiteness. We define

$$f_\phi = f^\mu|_{\mathbb{D}}\,.$$

It is a standard result in Teichmüller theory that f_ϕ is independent of the choice of quasiconformal extension w, and furthermore $f_\phi = f_\psi$ if and only if $\phi = \psi$ [11], [12].

From now on, let Σ be a Riemann surface of genus g bordered by n curves homeomorphic to $\mathbb{S}^1$ where $n > 0$; here the Riemann surface is "bordered" in the sense of Ahlfors and Sario [1, II.1.3]. We will also assume that the boundary of Σ consists of n connected components homeomorphic to $\mathbb{S}^1$ in the relative topology inherited from $\overline{\Sigma}$ with respect to the border structure. When we say that Σ is of genus g we mean that Σ is biholomorphic to a subset Σ^B of a compact Riemann surface $\widetilde{\Sigma}$ of genus g in such a way that the complement of $\overline{\Sigma^B}$ in $\widetilde{\Sigma}$ consists of n disjoint open sets biholomorphic to $\mathbb{D}$. Equivalently, the double of Σ has genus $2g+n$. We will refer to such a Riemann surface as a "bordered surface of type (g,n)".

We will also frequently make use of a kind of chart on "collars" of the boundary. Let

$$\mathbb{A}_r = \{z \,:\, 1 < |z| < r\}.$$

Definition 2.4. Let Σ be a bordered Riemann surface of genus g bordered by n curves $\partial_i\Sigma$, $i = 1, \ldots, n$, homeomorphic to $\mathbb{S}^1$. A collar chart (ζ, A) of $\partial_i\Sigma$ is an open set $A \subset \Sigma$ and a map $\zeta : A \to \mathbb{A}_r$ for some $r > 1$ such that $\partial_i\Sigma$ is contained in the closure of A and $\partial A \cap (\partial_i\Sigma)^c$ is compactly contained in Σ.

For any such A, $\mathbb{A}_r$, and ζ, ζ has a homeomorphic extension to $A \cup \partial_i\Sigma$.

We call A a "collar" of $\partial_i\Sigma$. We will also allow collar charts into annuli $r < |z| < 1$ when convenient.

The important property of bordered Riemann surfaces of type (g,n) for our purposes is that for any such surface and any i a collar chart exists [19]. Furthermore, A, r and ζ can be chosen so that $\partial A \backslash \partial_i \Sigma$ is an analytic curve. In that case ζ has a homeomorphic extension to the closure of A, which takes $\overline{A}$ onto the closed annulus $\overline{\mathbb{A}}_r$.

We may now define WP-class quasisymmetries between boundary curves of bordered Riemann surfaces, as in [18].

Definition 2.5. Let Σ_1 and Σ_2 be bordered Riemann surfaces of type (g_i, n_i) respectively, and let C_1 and C_2 be boundary curves of Σ_1 and Σ_2 respectively. Let $\mathrm{QS}_{\mathrm{WP}}(C_1, C_2)$ denote the set of orientation-preserving homeomorphisms $\phi : C_1 \to C_2$ such that there are collared charts H_i of C_i, $i = 1, 2$ respectively, and such that $\left. H_2 \circ \phi \circ H_1^{-1} \right|_{S^1} \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \mathbb{S}^1)$.

Equivalently, for *any* pair of collar charts H_i of C_i, $i = 1, 2$ respectively, $\left. H_2 \circ \phi \circ H_1^{-1} \right|_{\mathbb{S}^1} \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ [18].

Remark 2.6. The notation $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, C_1)$ will always be understood to refer to $\mathbb{S}^1$ as the boundary of an annulus $\mathbb{A}_r$ for $r > 1$. We will also write $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1) = \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \mathbb{S}^1)$.

3. WP-class Teichmüller space

3.1. **Quasiconformal maps with Beltrami differentials in $L^2_{-1,1}(\Sigma)$.** First, we make the following definition.

Definition 3.1. Let
$$\mathrm{TBD}(\Sigma) = L^\infty_{-1,1}(\Sigma) \cap L^2_{-1,1}(\Sigma)$$
and
$$\mathrm{BD}(\Sigma) = \{\mu \in \mathrm{TBD}(\Sigma) \,:\, \|\mu\|_{\infty,\Sigma} \leq K \ \text{ for some } \ K < 1\}.$$

The "BD" in the above notation stands for "Beltrami differentials". The "T" stands for "tangent". Analytically the notation "TBD" is slightly inaccurate, since $\mathrm{BD}(\Sigma)$ is not a Hilbert or Banach linear space or manifold, so that it does not have a tangent space in the standard sense. Nevertheless the notation distinguishes the spaces conveniently in terms of their upcoming roles.

Next we define the Weil-Petersson class Teichmüller space of bordered Riemann surfaces of type (g, n) (henceforth, we will abbreviate "Weil-Petersson" by "WP").

Definition 3.2. Let Σ and Σ_1 be bordered Riemann surfaces of type (g, n). Let $\mathrm{QC}_r(\Sigma, \Sigma_1)$ denote the set of quasiconformal maps $f : \Sigma \to \Sigma_1$ such that $\mu(f) \in \mathrm{BD}(\Sigma)$.

Definition 3.3 (WP-class Teichmüller space)**.** Let Σ be a bordered Riemann surface of type (g, n). We define an equivalence relation on triples (Σ, f, Σ') for $f : \Sigma \to \Sigma'$ quasiconformal as follows: $(\Sigma, f_1, \Sigma_1) \sim (\Sigma, f_2, \Sigma_2)$ if and only if there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary.

The WP-class Teichmüller space of Σ [18] is the set
$$T_{\mathrm{WP}}(\Sigma) = \{(\Sigma, f, \Sigma_1) \,:\, f \in \mathrm{QC}_r(\Sigma, \Sigma_1)\}/\sim .$$

The term "rel boundary" means that the homotopy is the identity on the boundary curve throughout.

Remark 3.4. In [18] the authors used a different definition of $T_{WP}(\Sigma)$; in place of $\mathrm{QC}_r(\Sigma, \Sigma_1)$, we use $\mathrm{QC}_0(\Sigma, \Sigma_1) = \{f \text{ quasiconformal} \,:\, f|_{\partial_i \Sigma} \in \mathrm{QS}_{\mathrm{WP}}(\partial_i \Sigma, \partial_i \Sigma_1)\}$. The definitions are in fact equivalent: any element of $\mathrm{QC}_0(\Sigma, \Sigma_1)$ is homotopic rel boundary to an element of

$\mathrm{QC}_r(\Sigma,\Sigma_1)$ [19]. Furthermore, any collection of maps $\phi_i \in \mathrm{QS}_{\mathrm{WP}}(\partial_i\Sigma,\partial_i\Sigma_1)$ has a quasiconformal extension in $\mathrm{QC}_r(\Sigma,\Sigma_1)$ [19].

Remark 3.5. In [18, Proposition 2.19] we showed that QC_0 is closed under composition. That is, if $f \in \mathrm{QC}_0(\Sigma_1,\Sigma_2)$ and $g \in \mathrm{QC}_0(\Sigma_2,\Sigma_3)$ then $g \circ f \in \mathrm{QC}_0(\Sigma_1,\Sigma_3)$. It would be satisfactory if this were also true for QC_r (especially for the theory of mapping class groups), although we do not yet see any independent reason why this should be true. Note that we do not need that result in this paper.

The complex Hilbert manifold structure on $T_{\mathrm{WP}}(\Sigma)$ is constructed using a natural fiber structure. This fiber structure is apparent in a closely related "rigged Teichmüller space" which we now define. We will use this ahead to define the coordinates on $T_{\mathrm{WP}}(\Sigma)$.

We say that Σ^P is a punctured Riemann surface of type (g,n) if it is a genus g Riemann surfaces with n points removed. The removed points are furthermore given a specific order $p_1,\ldots,p_n$, so the term "punctured Riemann surface of type (g,n)" refers to such a surface together with the ordering. For the purposes of this paper we could also think of the surfaces as compact Riemann surfaces with n distinguished points listed in a specific order; we will move between these two pictures freely without changing the notation.

Definition 3.6 (WP-class riggings)**.** Let Σ^P be a punctured Riemann surface of type (g,n) with punctures $p_1,\ldots,p_n$. A WP-class rigging on Σ^P is an n-tuple of maps $\phi=(\phi_1,\ldots,\phi_n)$ such that for each $i=1,\ldots,n$,

(1) $\phi_i:\mathbb{D}\to\Sigma^P$ is one-to-one and holomorphic,
(2) $\phi_i(0)=p_i$,
(3) ϕ_i has a quasiconformal extension to a neighbourhood of the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$,
(4) if $\zeta_i:U_i\to\mathbb{C}$ is a local biholomorphic coordinate on an open set $U_i\subset\Sigma^P$ containing $f_i(\overline{\mathbb{D}})$ such that $\zeta_i(p_i)=0$, then $\zeta_i\circ\phi_i\in\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, and
(5) Whenever $i\neq j$, $\phi_i(\overline{\mathbb{D}})\cap\phi_j(\overline{\mathbb{D}})$ is empty.

Denote the set of WP-class riggings on Σ^P by $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$.

When Σ^P and Σ^P_1 are Riemann surfaces of type (g,n), then if $f:\Sigma^P\to\Sigma^P_1$ is quasiconformal, it must extend to the compactification in such a way that it takes punctures to punctures.

Definition 3.7 (WP-class rigged Teichmüller space)**.** Let Σ^P be a punctured Riemann surface of type (g,n). The WP-class rigged Teichmüller space of Σ^P is the set

$$\widetilde{T}_{\mathrm{WP}}(\Sigma^P)=\{(\Sigma^P,f,\Sigma^P_1,\phi)\}/\sim$$

of equivalence classes $[\Sigma^P,f,\Sigma^P_1,\phi]$ where $f:\Sigma^P\to\Sigma^P_1$ is a quasiconformal map and $\phi\in\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P_1)$. The equivalence relation $\sim$ is defined as follows: $(\Sigma^P,f_1,\Sigma^P_1,\phi)\sim(\Sigma^P,f_2,\Sigma^P_2,\psi)$ if and only if there is a biholomorphism $\sigma:\Sigma^P\to\Sigma^P_1$ such that

(1) $f_2^{-1}\circ\sigma\circ f_1$ is homotopic to the identity rel boundary
(2) $\psi_i=\sigma\circ\phi_i$ for $i=1,\ldots,n$.

Remark 3.8. In this case, the requirement that $f_2^{-1}\circ\sigma\circ f_1$ is homotopic to the identity rel boundary implies that $f_2^{-1}\circ\sigma\circ f_1(p_i)=p_i$ (in the sense that the unique extensions of f_1, f_2 and σ satisfy this equality) and that the punctures are preserved throughout the homotopy.

Remark 3.9. In particular, if $(\Sigma^P, f_1, \Sigma_1^P, \phi) \sim (\Sigma^P, f_2, \Sigma_2^P, \psi)$ in $\widetilde{T}_0(\Sigma^P)$ then $(\Sigma^P, f_1, \Sigma_1^P) \sim (\Sigma^P, f_2, \Sigma_2^P)$ in the Teichmüller space $T(\Sigma^P)$, and $\sigma : \Sigma_1^P \to \Sigma_2^P$ preserves the ordering of the punctures.

Remark 3.10. In [18] we showed that $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$ has a complex Hilbert manifold structure in the case that $2g-2+n>0$. In this paper we will only consider $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$ for $2g-2+n>0$.

Next, we show that the WP-class Teichmüller space of bordered surfaces and the WP-class rigged Teichmüller space of punctured Riemann surfaces are closely related. The relation is obtained by "sewing caps" onto a given bordered Riemann surface to obtained a punctured surface. We outline this procedure below.

Definition 3.11. A parametrization $\tau_i : \mathbb{S}^1 \to \partial_i \Sigma$ of the ith boundary curve of a bordered Riemann surface is called analytic if there exists a collar chart ζ_i such that $\zeta_i \circ \tau_i$ is an analytic diffeomorphism of $\mathbb{S}^1$.

Let Σ be a bordered Riemann surface of type (g,n). Let $\tau = (\tau_1, \ldots, \tau_n)$ be a WP-class quasisymmetric parameterization of the boundary; that is

$$\tau_i \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \partial_i\Sigma), \quad i = 1, \ldots, n.$$

The existence of such a τ is not restrictive, as the following theorem shows. In fact, we show that every boundary curve can be analytically parameterized.

Theorem 3.12. *There is a parametrization $\tau = (\tau_1, \ldots, \tau_n)$ of $\partial\Sigma$ such that τ_i is analytic for each i. In particular, $\tau \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \partial\Sigma)$.*

Proof. Let (ζ_i, U_i) be a collar chart of $\partial_i\Sigma$. By the Schwarz reflection principle, ζ_i extends to an open neighbourhood of the boundary $\partial_i\Sigma$ in the double Σ^D of the Riemann surface Σ, which maps $\partial_i\Sigma$ into the circle $\mathbb{S}^1$. Let τ_i be the restriction to $\mathbb{S}^1$ of the extension of ζ_i^{-1}. Then $\zeta_i \circ \tau_i(z) = z$ satisfies the requirement. □

Given such a τ, the τ_i's are in particular quasisymmetries. Thus one can form the Riemann surface $\Sigma^P = \Sigma \#_\tau \cup_{i=1}^n \mathbb{D}$ which is defined as follows. We identify points p on the boundary of the nth disk with points q on the nth boundary curves, $p \sim q$, if $q = \tau_i(p)$. The resulting set

$$\Sigma^P = (\Sigma \sqcup \mathbb{D} \sqcup \cdots \sqcup \mathbb{D}) / \sim$$

is a topological space with the quotient topology and has a unique complex structure which agrees with the complex structures on Σ and each of the discs $\mathbb{D}$ [14, Theorems 3.2, 3.3]. We refer to Σ^P as being obtained from Σ by "sewing on caps via τ".

The inclusion maps from each disk $\mathbb{D}$ into Σ^P are holomorphic. Denote the resulting maps by $\tilde{\tau}_i : \mathbb{D} \to \Sigma^P$ and $\tilde{\tau} = (\tilde{\tau}_1, \ldots \tilde{\tau}_n)$. By [18, Proposition 4.7] $\tilde{\tau} \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$.

Now let $f \in \mathrm{QC}_r(\Sigma, \Sigma_1)$. Let Σ_1^P be obtained from Σ_1 by sewing on caps via $f \circ \tau = (f \circ \tau_1, \ldots, f \circ \tau_n)$. Note that $f \circ \tau_i \in \mathrm{QS}_{\mathrm{WP}}(\partial_i\Sigma, \partial_i\Sigma_1)$ for each i by [18, Proposition 4.8]. There is a natural extension of f to a map $\tilde{f} : \Sigma^P \to \Sigma_1^P$ defined as follows:

$$\tilde{f}(z) = \begin{cases} f(z), & z \in \overline{\Sigma} \\ z, & z \in \mathbb{D} \cup \cdots \cup \mathbb{D}. \end{cases} \tag{3.1}$$

We thus have a natural map from $T_{\mathrm{WP}}(\Sigma)$ into $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$.

$$\begin{aligned} \Pi : T_{\mathrm{WP}}(\Sigma) &\longrightarrow \widetilde{T}_{\mathrm{WP}}(\Sigma^P) \\ (\Sigma, f, \Sigma_1) &\longmapsto (\Sigma^P, \tilde{f}, \Sigma_1^P, \tilde{f} \circ \tilde{\tau}). \end{aligned} \tag{3.2}$$

Note that $\tilde{f} \circ \tilde{\tau}_i$ is holomorphic on each copy of the disk $\mathbb{D}$.

The map Π was used in [18] to construct an atlas on $T_{\mathrm{WP}}(\Sigma)$, making it a complex Hilbert manifold. We will describe the atlas of charts in the next section.

Remark 3.13. Observe that $\tilde{f}$ is the unique quasiconformal map (up to composition with a conformal map) solving the Beltrami equation on Σ^P with the Beltrami differential

$$\tilde{\mu}(z) = \begin{cases} \mu(f)(z), & z \in \Sigma \\ 0, & z \in \mathbb{D} \cup \cdots \cup \mathbb{D}. \end{cases}$$

Thus we may see the map Π as generalizing the Bers trick associating a conformal map f of the disk with elements of the universal Teichmüller space.

Finally we will need the following lemma.

Lemma 3.14. *Let $\tau \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \partial\Sigma)$ be an analytic parametrization of $\partial\Sigma$. Let Σ^P be obtained from Σ by sewing on caps via τ, and let $\tilde{\tau} : \mathbb{D} \to \Sigma^P$ be the resulting extension of τ. For each $i = 1, \ldots, n$ there is a chart (ζ_i, U_i) such that U_i contains the closure of $\tilde{\tau}_i(\mathbb{D})$ and $\zeta_i \circ \tilde{\tau}_i$ is the identity map on $\mathbb{S}^1$.*

Proof. Let (ξ_i, W_i) be any coordinate chart such that $\overline{\tilde{\tau}_i(\mathbb{D})} \subset W_i$. By assumption $\xi_i \circ \tilde{\tau}_i(\mathbb{S}^1)$ is an analytic curve in $\mathbb{C}$, so the Riemann map $\eta_i : \mathbb{D} \to \xi_i(\tilde{\tau}_i(\mathbb{D}))$ has a one-to-one holomorphic extension to some disk $\mathbb{D}_R = \{z : |z| < R\}$ for $R > 1$ onto a neighbourhood of the closure of $\xi_i(\tilde{\tau}_i(\mathbb{D}))$. By composing η_i by a Möbius transformation we can assume that $\eta_i^{-1} \circ \xi_i \circ \tilde{\tau}_i$ is the identity (since it maps the disk to the disk). Set $\zeta_i = \eta_i^{-1} \circ \xi_i$; this is a chart on some domain $U_i \subseteq W_i$ where U_i contains the closure of $\tilde{\tau}_i(\mathbb{D})$. ☐

3.2. **Gardiner-Schiffer coordinates and the complex structure.** In this section we define Gardiner-Schiffer coordinates and the complex structure on $T_{\mathrm{WP}}(\Sigma)$. Although the geometric idea is straightforward, the construction and rigorous proofs are somewhat involved. We restrict ourselves here to summarizing the necessary facts and refer the reader to [18] for a full treatment. In order to make rigorous statements about holomorphicity, we require the framework of marked holomorphic families of Earle and Fowler [3]. The appendix ahead contains a brief summary of the necessary elements of this theory for readers not familiar with it.

Gardiner-Schiffer variation is a technique for constructing coordinates on the Teichmüller space of a compact surface with punctures. See [5] or [12].

Let Σ^P be a punctured Riemann surface of type (g, n) and $[\Sigma^P, f, \Sigma_1^P] \in T(\Sigma^P)$. Let $d = 3g-3+n$ which is the dimension of the Teichmüller space $T(\Sigma^P)$. Let $D = (D_1, \ldots, D_d)$ be a d-tuple of disjoint open sets $D_i \subset \Sigma_1^P$, each of which is biholomorphic to the unit disk $\mathbb{D}$ via a map $\eta_i : D_i \to \mathbb{D}$. Let $w_\epsilon = z + \epsilon\bar{z}$, where $\epsilon \in \mathbb{C}$. For $|\epsilon|$ sufficiently small, w_ϵ is a quasiconformal homeomorphism and we let $D_\epsilon = w_\epsilon(\mathbb{D})$.

Let Ω be a small open connected neighbourhood of $0 \in \mathbb{C}^d$, and let $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \Omega$. By excising the disks D_i, and gluing in D_{ϵ_i} using the boundary identification $w_{\epsilon_i} \circ \eta_i$, one obtains a new Riemann surface $\Sigma_{1,\epsilon}^P$ and a quasiconformal map

$$\nu_\epsilon : \Sigma_1^P \longrightarrow \Sigma_{1,\epsilon}^P$$

defined by $\nu_\epsilon(z) = w_{\epsilon_i} \circ \eta_i$ for $z \in D_i$ and $\nu_\epsilon(z) = z$ for $z \in \Sigma_1^P \setminus D$. Note that ν_ϵ is holomorphic outside of the disks D_i, and on D_i the Beltrami differential of ν_ϵ is $\epsilon_i d\bar{z}/dz$ in the coordinate system η_i. We now define the Schiffer variation map

$$\begin{aligned} \mathfrak{S} : \Omega &\longrightarrow T(\Sigma^P) \\ \epsilon &\longmapsto [\Sigma^P, \nu_\epsilon \circ f, \Sigma_{1,\epsilon}^P] \end{aligned}$$

It follows directly from the construction that $\mathfrak{S}$ is holomorphic, but it can in fact give analytic coordinates on $T(\Sigma^P)$ as the following theorem from [5] or [12] shows.

Theorem 3.15. *Let $[\Sigma^P, f, \Sigma_1^P] \in T(\Sigma^P)$. Let $D_i \subset \Sigma_1^P, i = 1, \ldots, d$, be any disjoint biholomorphic images of the unit disk $\mathbb{D}$. There exist coordinates $\eta_i : D_i \to \mathbb{D}$ for $i = 1, \ldots, d$, and an open connected neighbourhood Ω of $0 \in \mathbb{C}^d$, such that $\mathfrak{S}$ is a biholomorphism onto its image (i.e. the inverse is a local coordinate chart).*

The collection of Riemann surfaces $\Sigma_{1,\epsilon}^P$ form a marked holomorphic family of Riemann surfaces (see Definition 5.3 ahead) as follows . By [18, Theorem 3.15] the set

$$S(\Omega, D) = \{(\epsilon, x) \,:\, \epsilon \in \Omega \text{ and } x \in \Sigma_{1,\epsilon}^P\}$$

is a complex manifold, with projection

$$\begin{aligned} \pi : S(\Omega, D) &\longrightarrow \Omega \\ (\epsilon, x) &\longmapsto \epsilon \end{aligned}$$

and strong global trivialization

$$\begin{aligned} \theta : \Omega \times \Sigma^P &\longrightarrow S(\Omega, D) \\ (\epsilon, q) &\longmapsto (\epsilon, \nu_\epsilon \circ f(q)) \end{aligned} \tag{3.3}$$

giving $S(\Omega, D)$ the structure of a marked holomorphic family of Riemann surfaces. In particular, note that $\Sigma_{1,\epsilon}^P$ is a complex submanifold of $S(\Omega, D)$ and for each fixed ϵ, $x \mapsto \theta(\epsilon, x)$ is a quasiconformal map from Σ^P to $\pi^{-1}(\epsilon) = \Sigma_{1,\epsilon}^P$

Using Theorem 5.8 we can embed the marked Schiffer family into the universal Teichmüller curve as described in the following theorem. Let $\mu(\epsilon) = \mu(\nu^\epsilon \circ f)$. Then the fiber in the Teichmüller curve over $\mathfrak{S}(\epsilon)$ is the canonical Riemann surface $\Sigma_{\mu(\epsilon)}^P$ as defined in (5.1). Note that $\Sigma_{1,\epsilon}^P$ and $\Sigma_{\mu(\epsilon)}^P$ are biholomorphically equivalent.

Theorem 3.16. *Let Σ^P be a punctured Riemann surface of type (g, n) such that $2g-2+n > 0$. Let $S(\Omega, D)$ be a marked Schiffer family as defined above. There is a morphism of marked holomorphic families (α, β) from $\pi : S(\Omega, D) \to \Omega$ to $\pi_T : \mathcal{T}(\Sigma^P) \to T(\Sigma^P)$ and moreover*

(1) *$\alpha(\epsilon) = \mathfrak{S}(\epsilon) = [\Sigma^P, \nu_\epsilon \circ f, \Sigma_{1,\epsilon}^P]$, and*
(2) *α and β are injective.*

Setting

$$\sigma_\epsilon(z) = \beta(\epsilon, z) : \Sigma_{1,\epsilon}^P \to \Sigma_{\mu(\epsilon)}^P$$

we have the biholomorphism

$$\begin{aligned}(3.4)\qquad \Gamma : S(\Omega, D) &\longrightarrow \pi_T^{-1}(\mathfrak{S}(\Omega)) \subseteq \mathcal{T}(\Sigma^P)\\ (\epsilon, p) &\longmapsto (\alpha, \beta)(\epsilon, p) = \left([\Sigma^P, \nu_\epsilon \circ f, \Sigma^P_{1,\epsilon}], \sigma_\epsilon(p)\right).\end{aligned}$$

Proof. Since the spaces are finite-dimensional, injective holomorphic functions are necessarily biholomorphic. So the only part not following immediately from Theorem 5.8 is the injectivity of α and β. Since $\mathfrak{S}$ is injective by Theorem 3.15, it remains only to show that β is injective. This follows from the fact that β is injective fiberwise and $\alpha \circ \pi = \pi_T \circ \beta$. □

Next, we need to define a local trivialization of the elements of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$.

Definition 3.17. Let Σ^P be a punctured Riemann surface of type (g, n). An n-chart (ζ, E) on Σ^P is a collection of open subsets $E = (E_1, \dots, E_n)$ of the compactification of Σ^P and local biholomorphic parameters $\zeta_i : E_i \to \mathbb{C}$ such that

(1) $E_i \cap E_j$ is empty whenever $i \neq j$,
(2) $p_i \in E_i$ for $i = 1, \dots, n$, and
(3) $\zeta_i(p_i) = 0$ for $i = 1, \dots, n$.

Definition 3.18. Let Σ^P be a punctured Riemann surface of type (g, n). Let (ζ, E) be an n-chart on Σ^P. We say that a collection $U_1, \dots, U_n$ of open sets $U_i \subseteq \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ is compatible with (ζ, E) if for all $f_i \in U_i$, $\overline{f_i(\mathbb{D})} \subseteq \zeta_i(E_i)$. In this case we will also say that $U = U_1 \times \cdots \times U_n \subset \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ is compatible with (ζ, E).

The existence of such open sets is not immediately obvious. By [17, Theorem 3.4], for any open subsets F_i of $\zeta_i(E_i)$, $i = 1, \dots, n$,

$$\{f_i \,:\, \overline{f_i(\mathbb{D})} \subseteq F_i\}$$

is open in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ and thus for example

$$\{(f_1, \dots, f_n) \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \,:\, \overline{f_i(\mathbb{D})} \subseteq F_i,\ i = 1, \dots, n\}$$

is open in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

Definition 3.19. Let $\mathfrak{S} : \Omega \to T(\Sigma^P)$ be Schiffer coordinates based at $[\Sigma^P, f, \Sigma^P_1]$. We say that an n-chart (ζ, E) on Σ^P_1 is compatible with $\mathfrak{S}$ if the closure of each disk D_i, $i = 1, \dots, d$, in the Schiffer variation is disjoint from the closure of each open set E_i in the n-chart.

This definition ensures that the Schiffer variation maps ν_ϵ are conformal on the closures of each n-chart; this is crucial for the construction of coordinates on $\widetilde{T}_{\mathrm{WP}}(\Sigma)$.

Definition 3.20. Let $\mathfrak{S} : \Omega \to T(\Sigma^P)$ be Schiffer coordinates based at $[\Sigma^P, f, \Sigma^P_1]$ corresponding to disks $D = (D_1, \dots, D_d)$, and let $S = S(\Omega, D)$ be the corresponding Schiffer family. Let (ζ, E) be an n-chart on Σ^P_1 and assume that $\mathfrak{S}$ is compatible with an n-chart (ζ, E). Let U be an open subset of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ which is compatible with (ζ, E). Define $F(U, S, \Omega)$ by

$$F(U, S, \Omega) = \{(\Sigma^P, \nu_\epsilon \circ f, \Sigma^P_{1,\epsilon}, \psi) \,:\, \psi = \nu_\epsilon \circ \zeta^{-1} \circ \phi,\ \phi \in U,\ \epsilon \in \Omega\}$$

where $\zeta^{-1} \circ \phi = (\zeta_1^{-1} \circ \phi_1, \dots, \zeta_n^{-1} \circ \phi_n)$.

It was shown in [18] that these define a base for a topology on $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$, and furthermore that the set of charts

$$\begin{aligned} \mathcal{G} : F(U,S,\Omega) &\longrightarrow \Omega \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \\ (\Sigma^P, \nu_\epsilon \circ f, \Sigma^P_{1,\epsilon}, \psi) &\longmapsto (\epsilon, \zeta \circ \nu_\epsilon^{-1} \circ \psi) \end{aligned} \tag{3.5}$$

is an atlas defining a complex Hilbert manifold structure on $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$.

Remark 3.21. In [18], we used the notation G for $\mathcal{G}^{-1}$.

The complex structure on $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$ passes upwards to $T_{\mathrm{WP}}(\Sigma)$, according to the following theorem which the authors proved in [18, Theorem 4.18].

Theorem 3.22. *If Σ is a bordered surface of type (g,n), with $2g-2+n>0$, then the collection of charts on $T_{\mathrm{WP}}(\Sigma)$ of the form $\mathcal{G}\circ\Pi$ is an atlas for a complex structure. Thus $T_{\mathrm{WP}}(\Sigma)$ is a complex Hilbert manifold. The map Π defined by (3.2) is a local biholomorphism; that is, for any point $p \in T_{\mathrm{WP}}(\Sigma)$ there is an open neighbourhood U of p such that the restriction of Π to U is a biholomorphism onto its image.*

We also have a natural projection from $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$ onto $T(\Sigma^P)$, given by

$$\begin{aligned} \mathcal{F} : \widetilde{T}_{\mathrm{WP}}(\Sigma^P) &\longrightarrow T(\Sigma^P) \\ ([\Sigma^P, f, \Sigma^P_1], \psi) &\longmapsto [\Sigma^P, f, \Sigma^P_1]. \end{aligned} \tag{3.6}$$

3.3. **A preparation theorem.** The purpose of this section is to use the sewing technology to prove the following fact: given a holomorphic curve through the identity in $T_{\mathrm{WP}}(\Sigma)$, it is possible to choose a Teichmüller equivalent curve in $\mathrm{BD}(\Sigma)$. That is, the curve can be chosen so that at each point the representative of the Teichmüller equivalence class has a Beltrami differential simultaneously in $L^2_{-1,1}(\Sigma)$ and $L^\infty_{-1,1}(\Sigma)$ (Theorem 3.25 ahead).

To prove this, we first need the following modification of a lemma of Takhtajan and Teo [22]. Denote by $\mathcal{S}(\phi)$ the Schwarzian of ϕ and by $\mathcal{A}(\phi)$ the pre-Schwarzian.

Lemma 3.23. *There is an open neighbourhood of $0 \in A^2_1(\mathbb{D})$ such that the map*

$$\begin{aligned} \Psi : A^2_1(\mathbb{D}) &\longrightarrow A^2_2(\mathbb{D}) \oplus \mathbb{C} \\ \psi &\longmapsto \left(\psi_z - \frac{1}{2}\psi^2, \psi(0)\right) \end{aligned}$$

is a biholomorphism onto its image.

In particular, there exists an open ball A centred on 0 in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ on which

$$\|\mathcal{S}(\phi)\|^2_{2,\mathbb{D}} + |\mathcal{A}(\phi)(0)|^2 \approx \|\mathcal{A}(\phi)\|^2_{2,\mathbb{D}}. \tag{3.7}$$

The norms are determined by treating $\mathcal{S}(\phi)$ as a $(2,0)$-differential and $\mathcal{A}(\phi)$ as a $(1,0)$-differential.

Proof. The map Ψ is in fact holomorphic and injective on $A^2_1(\mathbb{D})$ by [22, Lemma A.1]. In particular, $D\Psi|_0$ is injective and bounded. We claim that $D\Psi|_0$ is also surjective.

We will use the following approach. Let $(h,\alpha) \in A^2_2(\mathbb{D}) \oplus \mathbb{C}$. Let ϕ_t be the unique holomorphic function on $\mathbb{D}$ satisfying $\phi_t(0) = 0$, $\phi_t'(0) = 1$, $\mathcal{A}(\phi_t)(0) = t\alpha$ and $\mathcal{S}(\phi_t(z)) =$

$th(z)$. We will show that for some $\delta > 0$, if $|t| < \delta$ then $(\mathcal{S}(\phi_t)(z), \mathcal{A}(\phi_t)(0))$ is in the image of Ψ. This will show that $D\Psi|_0$ is surjective, and thus by the open mapping theorem for bounded linear maps it would follow that $D\Psi|_0$ is a topological isomorphism. Thus by the inverse function theorem the inverse is holomorphic [9]. Now if t is small enough then $\phi_t(z) \in \mathcal{O}^{\mathrm{qc}}$. To see this, observe first that by holomorphicity of inclusion $A_2^2(\mathbb{D}) \hookrightarrow A_2^\infty(\mathbb{D})$ [22] if $\|\mathcal{S}(\phi_t)\|_{2,\mathbb{D}}$ is small enough then by the classical criterion for quasiconformal extendibility, $\phi_t(z)$ has a quasiconformal extension to $\mathbb{C}$. By [22, Theorem 1.12, Part II], this together with the bound on $\|\mathcal{S}(\phi_t)\|_{2,\mathbb{D}}$ shows that $\mathcal{A}(\phi_t) \in A_1^2(\mathbb{D})$. This completes the proof of the first claim. The final claim follows from the scale invariance of $\mathcal{S}(\phi)$ and $\mathcal{A}(\phi)$. □

We will also need the following lemma due to the authors, explicitly separating out the contribution of the collar to the L^p norm.

Lemma 3.24 ([19])**.** *Let Σ be a bordered Riemann surface of genus g with n boundary curves. Fix $p \in [1, \infty)$. Let (ζ, U) be a collection of collar charts (ζ_i, U_i) into $\mathbb{D}$ for each boundary $i = 1, \ldots, n$. There exist annuli $\mathbb{A}_{r_i,1} = \{z : r_i < |z| < 1\} \subset \zeta_i(U_i)$ such that $|z| = r_i$ is compactly contained in $\zeta_i(U_i)$, a compact set M such that*

$$M \cup \zeta_1^{-1}(\mathbb{A}_{r_1,1}) \cup \cdots \cup \zeta_n^{-1}(\mathbb{A}_{r_n,1}) = \Sigma,$$

and constants a and b_i such that for any $\alpha \in L^p_{k,l}(\Sigma)$

$$\|\alpha\|_p \leq a\|\alpha\|_{\infty,M} + \sum_{i=1}^n b_i \left(\iint_{\mathbb{A}_{r_i,1}} \lambda_{\mathbb{D}}^{2-mp}(z) |\alpha_{U_i}(z)|^p \right)^{1/p}.$$

The constants b_i depend only on the collar charts (ζ_i, U_i), r_i, p, k and l (not on α), and a^p is the hyperbolic area of M.

Now we prove the central result of this section.

Theorem 3.25. *Let Σ be a bordered Riemann surface of type (g, n) such that $2g-2+n > 0$. Assume that $t \mapsto [\Sigma, f_t, \Sigma_t]$ is a holomorphic one-parameter curve in $T_{\mathrm{WP}}(\Sigma)$ such that $[\Sigma, f_0, \Sigma_0] = [\Sigma, Id, \Sigma]$. Then there exists $\delta > 0$ and representatives (Σ, f_t, Σ_t) so that the following properties are satisfied for $|t| < \delta$:*

1. *$\|\mu(f_t)\|_2$ is uniformly bounded in t, where $\mu(f_t)$ is the Beltrami differential of f_t on Σ,*
2. *$t \mapsto \mu(f_t)$ is a holomorphic curve in $L^2_{-1,1}(\Sigma)$, and*
3. *$t \mapsto \mu(f_t)$ is a holomorphic curve in $L^\infty_{-1,1}(\Sigma)$.*

Proof. As before, sew on caps via some $\tau \in \mathrm{QS}_{\mathrm{WP}}(\Sigma)$ to obtain the associated punctured surface Σ^P. By Theorem 3.22 every curve through $[\Sigma, \mathrm{Id}, \Sigma]$ is the inverse image under Π of a curve through $[\Sigma^P, \mathrm{Id}, \Sigma^P, \tilde{\tau}] \in \widetilde{T}_{\mathrm{WP}}(\Sigma)$. Thus it suffices to describe curves through this point in $\widetilde{T}_{\mathrm{WP}}(\Sigma)$. By [18, Corollary 4.22] the complex structure on $T_{\mathrm{WP}}(\Sigma)$ is independent of the choice of rigging τ used to sew on caps. Thus without loss of generality, we may assume that τ is an analytic parametrization (the existence of such a τ is guaranteed by Theorem 3.12). By Lemma 3.14 there is an n-chart (ζ, E) such that $\zeta_i \circ \tilde{\tau}_i$ is the identity on $\mathbb{D}$.

Fix a coordinate chart $\mathcal{G}$ on a neighbourhood $F(U, S, \Omega)$ of $([\Sigma^P, \mathrm{Id}, \Sigma^P], \tilde{\tau})$ in $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$, with associated compatible n-chart (ζ, E) on Σ^P. Let K_i be open, connected, simply-connected neighbourhoods of each puncture for $i = 1, \ldots, n$ such that $\overline{K_i} \subset E_i$ for $i =$

$1, \ldots, n$. We may choose the chart $F(U, S, \Omega)$ in Definition 3.20 above such that $\overline{\phi_i(\mathbb{D})} \subset \zeta_i(K_i)$ for all $\phi_i \in U$ and $i = 1, \ldots, n$.

Given the holomorphic curve $\lambda(t)$ in $T_{WP}(\Sigma)$, assume that it is in the image of some chart $\mathcal{G} \circ \Pi$ on $F(U, S, \Omega)$ of the above form (perhaps shrinking the t-domain of the curve). Let $(\epsilon(t), w_1(t,z), \ldots, w_n(t,z)) \in \Omega \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} = \mathcal{G} \circ \Pi(\lambda(t))$ with $w_i(0,z) = z$, $i = 1, \ldots, n$. By [22, Lemma 2.1, Page 22], one has

$$\sup_{\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}(w_i(t,\cdot))(z)| \leq \sqrt{\frac{12}{\pi}} \left\{ \iint_{\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}(w_i(t,\cdot))(z)|^2 \, dA \right\}^{\frac{1}{2}}. \tag{3.8}$$

Now using Lemma 3.23, there exists $C > 0$ such that for all $i = 1, \ldots, n$ one has

$$\left\{ \iint_{\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}(w_i(t,\cdot))(z)|^2 \, dA \right\}^{\frac{1}{2}} \leq C(|\mathcal{A}(w_i(t,\cdot))(0)| + \left\{ \iint_{\mathbb{D}} |\mathcal{A}(w_i(t,\cdot))(z)|^2 \, dA \right\}^{\frac{1}{2}}). \tag{3.9}$$

Letting $P_i(t) := |\mathcal{A}(w_i(t,\cdot))(0)| + \{\iint_{\mathbb{D}} |\mathcal{A}(w_i(t,\cdot))(z)|^2 \, dA\}^{\frac{1}{2}}$ we know that $\mathcal{A}(w_i(t,\cdot))(0)$ is continuous, and by the definition of the complex structure on T_{WP}, $\|\mathcal{A}(w_i(t,\cdot))\|_{L^2(\mathbb{D})}$ is also a continuous function of t. Therefore since $\mathcal{A}(w_i(0,\cdot)) = 0$ we have $P_i(0) = 0$ and hence there exists an $s > 0$ such that $P_i(t) \leq \sqrt{\pi/12C^2}$ for all $i = 1, \ldots, n$, if $|t| < s$. This fact together with (3.8) yields

$$\sup_{i=1,\ldots,n} \sup_{\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}(w_i(t,\cdot))(z)| \leq 1 \tag{3.10}$$

for $i = 1, \ldots, n$. Given (3.10), the Ahlfors-Weill reflection result [11, Theorem 5.1, Chpt II] yields that $w_i(t,z)$ has a jointly continuous quasiconformal extension to $\mathbb{C}$ with dilatation m^i_t satisfying

$$m^i_t(1/\bar{z}) = -\frac{(1-|z|^2)^2}{2} \frac{z^2}{\bar{z}^2} \mathcal{S}(w_i(t,\cdot))(z). \tag{3.11}$$

Thus for $|t| < s$

$$\iint_{\mathbb{D}} \frac{|m^i_t(1/\bar{z})|^2}{(1-|z|^2)^2} \, dA = \frac{1}{4} \iint_{\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}(w_i(t,\cdot))(z)|^2 \, dA. \tag{3.12}$$

Now using once again (3.9) and the bound on $P_i(t)$ obtained above, it is readily seen that there is an M such that $\|m^i_t\|^2_{2,\mathbb{D}^*} \leq M$ for each i and small enough t. This yields the uniform boundedness of $\|m^i_t\|_{2,\mathbb{D}^*}$ for $|t| < s$.

Choose simple closed analytic curves γ_i in $\zeta_i(E_i)$ so that γ_i and $\overline{K_i}$ are nested and non-intersecting; that is, γ_i encloses $\overline{K_i}$. Choose $R_i > 1$ such that $w_i(t, |z| = R_i)$ is enclosed by γ_i and does not intersect it; this can be done for all $|t| < s' < s$ for some s' since the extension $w_i(t,z)$ is jointly continuous. Let L_i be the doubly-connected region bounded by $\zeta_i^{-1}(|z| = R_i)$ and $\zeta_i^{-1}(\gamma_i)$. Let Y denote the pre-compact subset of Σ bounded by the curves $\zeta_i^{-1}(\gamma_i)$.

We will construct a family of quasiconformal maps $g_t : \Sigma^P \to \Sigma^P$ with the following properties.

(a) g_t is the identity on $\overline{Y}$.

(b) $\zeta_i \circ g_t \circ \zeta_i^{-1}$ is the Ahlfors-Weill extension of $w_i(t,z)$ on $|z| \leq R$ with dilatation given by (3.11).
(c) The Beltrami differential of $\zeta_i \circ g_t \circ \zeta_i^{-1}$ is holomorphic as a map from t into $L^\infty_{-1,1}$ on $\zeta_i(L_i)$.
(d) g_0 is the identity on Σ^P.

Observe that if g_t has these properties, then setting

$$F_t = \nu_{\epsilon(t)} \circ g_t : \Sigma^P \to \Sigma^P_t$$
$$f_t = F_t|_\Sigma : \Sigma \to \Sigma_t := F_t(\Sigma)$$

we have that

$$\mathcal{G} \circ \Pi([\Sigma, f_t, \Sigma_t]) = (\epsilon(t), w_1(t,\cdot), \ldots, w_n(t,\cdot))$$

and $(\Sigma, f_0, \Sigma_0) = (\Sigma, \mathrm{Id}, \Sigma)$. It follows that (Σ, f_t, Σ_t) is a representative of $\lambda(t)$. We will show momentarily that (Σ, f_t, Σ_t) has the properties (1), (2), and (3). However we must first establish the existence of g_t.

Denote the Ahlfors-Weill extension of $w_i(t,z)$ by $\hat{w}_i(t,z)$. It is well-known see e.g. [11], that the Ahlfors-Weill extension is holomorphic in t for fixed z so in particular the restriction of $t \mapsto \hat{w}_i(t,z)$ to $|z| \leq R_i$ is a holomorphic motion. By the extended lambda lemma [20], denoting by W the region enclosed by γ_i, there is a holomorphic motion $H_i : \Delta \times W \to W$ for some disc Δ centred on 0, which equals $\hat{w}_i(t,z)$ on $|z| \leq R$ and equals the identity on γ_i. Setting

$$g_t(p) = \begin{cases} \zeta_i^{-1} \circ H_i(t, \zeta_i(p)) & p \in \Sigma \backslash \overline{Y} \\ p & p \in \overline{Y} \end{cases}$$

we have that g_t satisfies properties (a) through (d).

Next, we show that (Σ, f_t, Σ_t) has the claimed properties. The uniform L^2 bound can be established easily as follows. Fix r_i such that $0 < r_i < R_i$ and let $\Gamma_i = \zeta_i^{-1}(|z| = r_i)$. Denote by V_i the collar neighbourhood bounded by $\partial_i \Sigma$ and Γ_i. The Beltrami differential of g_t satisfies $|\mu(g_t)(\zeta_i^{-1}(z))| = |\mu(\zeta_i \circ g_t \circ \zeta_i^{-1})(z)| = |m^i_t(1/\bar{z})|$, and we have shown that the L^2 norm of $m^i_t(1/\bar{z})$ is uniformly bounded on $|t| < s'$ and $z \in \mathbb{D}$. Thus $\|\mu(g_t)\|_{2,V_i}$ is uniformly bounded for $|t| < s'$ for all i. Now using the fact that g_t is the identity on $\overline{Y}$ and the Schiffer variation $\nu_{\epsilon(t)}$ has zero Beltrami differential outside of disks D_k disjoint from E_i, we see that

$$\mu(f_t) = \mu(\nu_{\epsilon(t)} \circ g_t) = \begin{cases} \epsilon(t) d\bar{z}/dz & z \in D_1 \cup \cdots \cup D_d \\ \mu(g_t) & z \in \Sigma \backslash Y \\ 0 & \text{otherwise.} \end{cases}$$

(Note that the expression on the Schiffer disks is in terms of the local parameter on each of those disks). Thus applying Lemma 3.24 with $k = -1$, $l = 1$ and $p = 2$ and with the collar chart $(\zeta_i|_{V_i}, V_i)$, and using the fact that the L^∞ norm of any Beltrami differential is bounded by one, we have a uniform bound on $\|\mu(f_t)\|_{2,\Sigma}$ for $|t| < s'$. This proves property (1) for $|t| < s'$.

We now prove the second and third claims. By assumption, each $w_i(t,z)$ is a holomorphic curve in $\mathcal{O}^{\mathrm{qc}}_{WP}(\mathbb{D})$. Therefore for any t_0 in a sufficiently small neighbourhood $|t| < s''$ of 0 there is a holomorphic function g_{t_0} on $\mathbb{D}$ such that

$$\lim_{t \to t_0} \iint_{\mathbb{D}} (1 - |z|^2)^2 \left| \frac{\mathcal{S}(w(t,z)) - \mathcal{S}(w(t_0,z))}{t - t_0} - g_{t_0}(z) \right|^2 dA = 0. \tag{3.13}$$

For the Ahlfors-Weill extension of $w(t,z)$ with dilatation (3.11), setting

$$\omega^i_{t_0}(1/\bar{z}) = -\frac{(1-|z|^2)^2}{2}\frac{z^2}{\bar{z}^2}\cdot g_{t_0}(z)$$

we then have that

$$\lim_{t\to t_0}\iint_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\left|\frac{\mu^i_t(1/\bar{z})-\mu^i_{t_0}(1/\bar{z})}{t-t_0} - \omega_{t_0}(1/\bar{z})\right|^2 = 0. \tag{3.14}$$

Observe also that since $A^2_2(\mathbb{D}) \hookrightarrow A^\infty_2(\mathbb{D})$ is a bounded inclusion [22], (3.13) also implies that

$$\lim_{t\to t_0}\left\|(1-|z|^2)^2\frac{\mathcal{S}(w_i(t,z))-\mathcal{S}(w_i(t_0,z))}{t-t_0} - g_{t_0}(z)\right\|_\infty = 0$$

and hence again by (3.11)

$$\lim_{t\to t_0}\left\|\frac{\mu^i_t(1/\bar{z})-\mu^i_{t_0}(1/\bar{z})}{t-t_0} - \omega_{t_0}(1/\bar{z})\right\|_\infty = 0. \tag{3.15}$$

We first prove that (Σ, f_t, Σ_t) has property (3). We need to establish that there is a Beltrami differential κ_{t_0} on Σ such that

$$\lim_{t\to t_0}\left\|\frac{\mu(g_t)-\mu(g_{t_0})}{t-t_0} - \kappa_{t_0}\right\|_\infty = 0 \tag{3.16}$$

holds almost everywhere. It is enough to show the existence of such a Beltrami differential on each portion of the Riemann surface individually. Using (3.15) on $1 < |z| < R_i$, and lifting ω_{t_0} to Σ via ζ_i^{-1} establishes the claim on the region bounded by $\zeta_i^{-1}(|z| = R_i)$ and $\partial_i\Sigma$, since the Beltrami differential of $\nu_{\epsilon(t)}$ is zero on this region. Property (c) of g_t establishes the claim on the region L_i, again using the fact that the Beltrami differential of $\nu_{\epsilon(t)}$ is zero there. Finally, on Y the claim follows from the fact that the Beltrami differential of g_t is zero there, and that in coordinates the Beltrami differential of $\nu_{\epsilon(t)}$ is just $\epsilon(t)d\bar{z}/dz$ on the Schiffer disks D_k and zero otherwise; note that $\epsilon(t)$ is a holomorphic function of t. This establishes property (3) for $|t| < s''$.

Next we establish property (2). We will again use Lemma 3.24 in the case that $k = -1$, $l = 1$ and $p = 2$. We again use the collar chart $(\zeta_i|_{V_i}, V_i)$. We then have the estimate (for some compact M and regions $\mathbb{A}_{r'_i}$ with $r'_i < r_i$)

$$\begin{aligned}\left\|\frac{\mu(f_t)-\mu(f_{t_0})}{t-t_0} - \omega_{t_0}\right\|_{2,\Sigma} &\leq a\left\|\frac{\mu(f_t)-\mu(f_{t_0})}{t-t_0} - \omega_{t_0}\right\|_{\infty,M} \\ &\quad + \sum_{i=1}^n b_i\left(\iint_{\mathbb{A}_{r'_i,1}} \lambda^2_{\mathbb{D}}(z)\left|\frac{\mu^i_t(1/\bar{z})-\mu^i_{t_0}(1/\bar{z})}{t-t_0} - \omega_{t_0}(1/\bar{z})\right|^2\right)^{1/2}.\end{aligned}$$

The first term goes to zero by property (3), and the remaining terms go to zero by (3.14). This establishes property (2) on $|t| < s''$. Now taking $\delta = \min(s', s'')$ proves the theorem. □

4. Tangent space to WP-class Teichmüller space and the WP metric

In this section, we demonstrate the convergence on refined Teichmüller space of the generalized Weil-Petersson metric.

4.1. **The tangent space to Teichmüller space.** First, we need some results on the function spaces which will serve as models of the tangent space. In all of the following, Σ will be a bordered Riemann surface of type (g,n).

Consider the spaces $L^2_{0,2}(\Sigma)$ and $L^\infty_{0,2}(\Sigma)$ of 2-differentials. We have a well-defined mapping from these spaces into $L^2_{-1,1}(\Sigma)$ and $L^\infty_{-1,1}(\Sigma)$ as follows. Let ψ be a two-differential, given in local coordinates (ζ, U) by $\psi_U(z)d\bar{z}^2$. Assume that the hyperbolic metric on Σ is given by $\rho_U(z)^2|dz|^2$ in local coordinates. It is easily checked that the locally defined functions

$$\psi_U(z)\rho_U^{-2}(z)$$

transform under change of coordinates as a $(-1,1)$ differential, and hence define a global $(-1,1)$ differential. Denote this map from 2-differentials to $(-1,1)$-differentials by $\mathfrak{B}$. It's not hard to check that $\mathfrak{B}$ has an inverse (obtained by multiplying by $\rho_U(z)^2$ in local coordinates). The following property of $\mathfrak{B}$ is an immediate consequence of Definition 2.1 and the definition of the norm.

Proposition 4.1. *Let Σ be a bordered Riemann surface of type (g,n) and let $\mathfrak{B}$ be defined as above. For any $p \in [1,\infty]$*

$$\mathfrak{B}(L^p_{0,2}(\Sigma)) = L^p_{-1,1}(\Sigma).$$

Furthermore, $\|\mathfrak{B}(\alpha)\|_p = \|\alpha\|_p$ for any $\alpha \in L^p_{0,2}(\Sigma)$.

Remark 4.2. This Proposition generalizes to other k, l by dividing by other powers of the hyperbolic metric, and also to arbitrary hyperbolic surfaces. However we do not need this here.

We now define the model spaces for the tangent space.

Definition 4.3. Let $\mathfrak{B}$ be as above. Let $\overline{A^i_2(\Sigma)}$ denote the set of complex conjugates of elements of $A^i_2(\Sigma)$ for $i = 2, \infty$. Let

$$\begin{aligned} H_{-1,1}(\Sigma) &= \mathfrak{B}\left(\overline{A^2_2(\Sigma)}\right) \\ \Omega_{-1,1}(\Sigma) &= \mathfrak{B}\left(\overline{A^\infty_2(\Sigma)}\right). \end{aligned}$$

Observe that $\mathfrak{B}$ is a bounded linear isomorphism in both cases, if $H_{-1,1}(\Sigma)$ and $\Omega_{-1,1}(\Sigma)$ are endowed with the norms inherited from $L^2_{-1,1}(\Sigma)$ and $L^\infty_{-1,1}(\Sigma)$ respectively.

For example, we have that

$$\begin{aligned} H_{-1,1}(\mathbb{D}^*) &= \left\{ (1-|z|^2)^2\overline{\psi(z)}d\bar{z}/dz \,:\, \iint_{\mathbb{D}^*} (1-|z|^2)^2|\psi(z)|^2\,dA < \infty \right\} \\ \Omega_{-1,1}(\mathbb{D}^*) &= \left\{ (1-|z|^2)^2\overline{\psi(z)}d\bar{z}/dz \,:\, \sup_{z\in\mathbb{D}^*}(1-|z|^2)^2|\psi(z)| < \infty \right\}. \end{aligned}$$

The space $\Omega_{-1,1}(\Sigma)$ is well-known to be complementary to the so-called infinitesimally trivial Beltrami differentials. It was shown by Takhtajan and Teo [22] that $H_{-1,1}(\mathbb{D}^*)$ is the tangent space to the WP-class universal Teichmüller space (although they do not use that term), and $T_{\mathrm{WP}}(\mathbb{D}^*)$ can be modelled by $H_{-1,1}(\mathbb{D}^*)$. Furthermore, the Weil-Petersson metric converges on this tangent space. We will show that this is true for $H_{-1,1}(\Sigma)$, if one uses the Hilbert manifold structure which the authors defined in [18].

Remark 4.4. Lifting to the cover $\mathbb{D}^*$ of Σ and attacking this problem using differentials which are invariant under the group of deck transformations does not appear to provide any advantage. This is because the relevant lifted differentials are L^2 on the fundamental domain but not on the cover $\mathbb{D}^*$. If the L^2 norm is non-zero on any fundamental domain, then the L^2 norm of the lift to $\mathbb{D}^*$ must be infinite. Thus $H_{-1,1}(\Sigma)$ cannot be expressed as a set of invariant differentials in $H_{-1,1}(\mathbb{D}^*)$. This is an unavoidable consequence of replacing the L^∞ norm with an L^2 norm.

Before proceeding, we must establish some analytic results. The main result which we require is the following.

Theorem 4.5. *If Σ is a bordered Riemann surface of type (g, n), then $H_{-1,1}(\Sigma) \subset \Omega_{-1,1}(\Sigma)$. Furthermore, the inclusion map is bounded.*

First, we require two lemmas.

Lemma 4.6. *Let $f : \mathbb{A}_r \to \mathbb{C}$ be a holomorphic function on $\mathbb{A}_r$. Then for any $t \in (1, r)$*

$$\sup_{z\in\mathbb{A}_t}(1-|z|^2)^2|f(z)| \leq C(r,t)\left(\iint_{\mathbb{A}_r}(1-|z|^2)^2|f(z)|^2\,dA\right)^{1/2} \tag{4.1}$$

with

$$C(r,t) = \frac{4}{\sqrt{2\pi}}\frac{\sqrt{r^2+t^2}}{r^2-t^2}\frac{(1-t^2)^2}{r^2+3} + 4\frac{\sqrt{3}}{\sqrt{\pi}}t.$$

Proof. To prove this lemma, let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent series of f in $\mathbb{A}_r$. An elementary calculation reveals that

$$\iint_{\mathbb{A}_r}|f(z)|^2(1-|z|^2)^2 dA = 2\pi\sum_{n=-\infty}^{\infty}|a_n|^2 I_n(r), \tag{4.2}$$

where $I_n(r) := \frac{1}{2}\int_1^{r^2}\rho^n(1-\rho)^2\,d\rho$. Now the Cauchy-Schwarz inequality yields

$$\begin{aligned}|f(z)| &\leq \left(\sum_{n=-\infty}^{\infty}|a_n|^2 I_n(r)\right)^{1/2}\left(\sum_{n=-\infty}^{\infty}\frac{|z|^{2n}}{I_n(r)}\right)^{1/2}\\ &= \frac{1}{\sqrt{2\pi}}\left(\iint_{\mathbb{A}_r}(1-|z|^2)^2|f(z)|^2\,dA\right)^{1/2}\left(\sum_{n=-\infty}^{\infty}\frac{|z|^{2n}}{I_n(r)}\right)^{\frac{1}{2}}. \qquad (4.3)\end{aligned}$$

Now let us estimate the quantity $\sum_{n=-\infty}^{\infty}\frac{|z|^{2n}}{I_n(r)}$. To this end we split the sum as follows

$$\sum_{n=0}^{\infty}\frac{|z|^{2n}}{I_n(r)} + \sum_{n=1}^{\infty}\frac{|z|^{-2n}}{I_{-n}(r)} := \mathbf{I} + \mathbf{J}. \tag{4.4}$$

We observe that since on $\mathbb{A}_r$ we have $1 < |z| < r$, then $0 < \frac{|z|^2-r^2}{1-r^2} < 1$ on $\mathbb{A}_r$. Bearing this fact in mind we proceed with the estimates of the above terms.

To estimate $\mathbf{I}$, take an s with $0 < s < \frac{|z|^2-r^2}{1-r^2}$ and set $r_s = s + (1-s)r^2$, for that choice of s. This yields that

(4.5)

$$I_n(r) \geq \frac{1}{2}\int_{r_s}^{r^2} \rho^n(\rho-1)^2\,d\rho \geq \frac{r_s^n}{2}(r_s-1)^2\int_{r_s}^{r^2} d\rho = \frac{r_s^n}{2}(r_s-1)^2(r^2-r_s) = \frac{r_s^n}{2}s(1-s)^2(r^2-1)^3.$$

Therefore since $0 < s < \frac{|z|^2-r^2}{1-r^2}$ implies that $\frac{|z|^2}{r_s} < 1$, we have

$$\mathbf{I} \leq \frac{2}{s(1-s)^2}\frac{1}{(r^2-1)^3}\sum_{n=0}^{\infty}\left(\frac{|z|^2}{r_s}\right)^n = \frac{2}{s(1-s)^2}\frac{1}{(r^2-1)^3}\frac{r_s}{r_s-|z|^2}. \tag{4.6}$$

Now we turn to the estimate for $\mathbf{J}$, to this end take an s' with $\frac{|z|^2-r^2}{1-r^2} < s' < 1$ and set $r_{s'} := s' + (1-s')r^2$ for that choice of s'. This yields that

$$I_{-n}(r) \geq \frac{1}{2}\int_1^{r_{s'}} \rho^{-n}(\rho-1)^2\,d\rho \geq \frac{r_{s'}^{-n}}{2}\int_1^{r_{s'}}(\rho-1)^2\,d\rho = \frac{r_{s'}^{-n}}{6}(1-s')^3(r^2-1)^3. \tag{4.7}$$

Hence since $\frac{|z|^2-r^2}{1-r^2} < s' < 1$ implies that $|z|^{-2}r_{s'} < 1$

$$\mathbf{J} \leq \frac{6}{(1-s')^3}\frac{1}{(r^2-1)^3}\sum_{n=0}^{\infty}(|z|^{-2}r_{s'})^n = \frac{6}{(1-s')^3}\frac{1}{(r^2-1)^3}\frac{|z|^2}{|z|^2-r_{s'}}. \tag{4.8}$$

Now (4.6) and (4.8) yield that

$$(1-|z|^2)^2\left(\sum_{n=-\infty}^{\infty}\frac{|z|^{2n}}{I_n(r)}\right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{\sqrt{s}(1-s)}\frac{(1-|z|^2)^2}{(r^2-1)^{3/2}}\frac{\sqrt{r_s}}{\sqrt{r_s-|z|^2}} + \frac{\sqrt{6}}{(1-s')^{3/2}}\frac{(1-|z|^2)^2}{(r^2-1)^{3/2}}\frac{|z|}{\sqrt{|z|^2-r_{s'}}} := R_1 + R_2. \tag{4.9}$$

Now since the inequality (4.9) is valid for all $s \in (0, \frac{|z|^2-r^2}{1-r^2})$ and $s' \in (\frac{|z|^2-r^2}{1-r^2}, 1)$, we take $s = \frac{|z|^2-r^2}{2(1-r^2)}$, $s' = \frac{1}{2} + \frac{|z|^2-r^2}{2(1-r^2)}$. With these choices of s and s' we have, $1-s = \frac{|z|^2+r^2-2}{2(r^2-1)}$, $r_s = \frac{r^2+|z|^2}{2}$, $r_s - |z|^2 = \frac{r^2-|z|^2}{2}$ and $1-s' = \frac{1}{2}(\frac{|z|^2-1}{r^2-1})$, $r_{s'} = \frac{1}{2}(1+|z|^2)$, $|z|^2 - r_{s'} = \frac{1}{2}(|z|^2-1)$. Plugging in these values into R_1 and R_2 yields that $R_1 = 4\frac{\sqrt{r^2+|z|^2}}{r^2-|z|^2}\frac{(1-|z|^2)^2}{|z|^2+r^2+2}$ and $R_2 = 4\sqrt{6}|z|$. This and (4.3) yield for $z \in \mathbb{A}_r$ the pointwise estimate

(4.10)

$$(1-|z|^2)^2|f(z)| \leq \left(\frac{4}{\sqrt{2\pi}}\frac{\sqrt{r^2+|z|^2}}{r^2-|z|^2}\frac{(1-|z|^2)^2}{|z|^2+r^2+2} + 4\frac{\sqrt{3}}{\sqrt{\pi}}|z|\right)\left(\iint_{\mathbb{A}_r}(1-|z|^2)^2|f(z)|^2\,dA\right)^{1/2}.$$

Now observing that for $1 < |z| < t$ one has

$$\frac{4}{\sqrt{2\pi}}\frac{\sqrt{r^2+|z|^2}}{r^2-|z|^2}\frac{(1-|z|^2)^2}{|z|^2+r^2+2} + 4\frac{\sqrt{3}}{\sqrt{\pi}}|z| \leq \frac{4}{\sqrt{2\pi}}\frac{\sqrt{r^2+t^2}}{r^2-t^2}\frac{(1-t^2)^2}{r^2+3} + 4\frac{\sqrt{3}}{\sqrt{\pi}}t,$$

taking the supremum in (4.10) over all $z \in \mathbb{A}_t$ yields the desired estimate. $\square$

Lemma 4.7. *Let Σ be a bordered Riemann surface of type (g,n) and let M be compactly contained in Σ. There is a constant D_M depending only on M such that for any $\alpha \in H_{-1,1}(\Sigma)$,*

$$\|\alpha\|_{\infty,M,\Sigma} \leq D_M \|\alpha\|_{2,\Sigma}.$$

Proof. Since M is compact in Σ, there is a compact subset N of Σ such that M is a subset of the interior of N. There is a finite collection of open sets W_k and V_k, $k = 1, \ldots, m$ such that

(1) $\overline{W_k} \subseteq V_k \subseteq N$ for $k = 1, \ldots, m$ where $\overline{W_k}$ denotes the closure of W_k,
(2) There are coordinate charts $\eta_k : V_k \to G_k$, where G_k are bounded, open connected subsets of $\mathbb{C}$, and
(3) $M \subseteq \cup_{k=1}^m W_k$.

Since N is a compact subset of Σ, and there are only finitely many charts (η_k, V_k), there is a constant $C > 0$ such that if ρ_{V_k} denotes the hyperbolic metric in local coordinates then

$$\frac{1}{C} \leq \rho_{V_k}(z) \leq C \tag{4.11}$$

for each $k = 1, \ldots, m$.

Now let $\alpha \in H_{-1,1}(\Sigma)$. There is a $\psi \in A_2^2(\Sigma)$ such that in η_k coordinates, α has the form $\rho_{V_k}(z)^{-2}\overline{\psi_{V_k}(z)} d\bar{z}/dz$ where $\psi_{V_k}(z)dz^2$ is the expression for ψ in η_k coordinates. For all $z \in W_k$, we have by an elementary estimate that there is a constant E_k, which is independent of ψ (depending only on $\eta_k(W_k)$ and $\eta_k(V_k)$) such that

$$|\psi_{V_k}(z)| \leq E_k \left(\iint_{\eta_k(V_k)} |\psi_{V_k}(z)|^2 \, dA \right)^{1/2}$$

where dA denotes the area element $d\bar{z} \wedge dz/2i$. This can be obtained by applying the mean value property of holomorphic functions. Thus, applying (4.11) twice, we see that

$$\begin{aligned}
\rho_{V_k}^{-2}(z)|\psi_{V_k}(z)| &\leq C^2 |\psi_{V_k}(z)| \\
&\leq C^2 E_k \left(\iint_{\eta_k(V_k)} |\psi_{V_k}(z)|^2 \, dA \right)^{1/2} \\
&\leq C^3 E_k \left(\iint_{\eta_k(V_k)} \rho_{V_k}(z)^{-2} |\psi_{V_k}(z)|^2 \, dA \right)^{1/2} \\
&\leq C^3 E_k \|\alpha\|_{2,\Sigma}.
\end{aligned}$$

Since W_k cover M, taking

$$D_M = C^3 \max\{E_1, \ldots, E_k\}$$

the claim is proven. ∎

We will also need the following Lemma proven in [19].

Lemma 4.8. *Let Σ be a bordered Riemann surface of type (g,n) and let (ζ_i, U_i) be a collar chart of $\partial_i \Sigma$. Let $\lambda_{\mathbb{D}}(z) = 1/(1-|z|^2)$ and let $\rho_{U_i}^2 |dz|^2$ be the expression for the hyperbolic metric in (ζ_i, U_i) coordinates. There is an annulus $\mathbb{A}_{r,1} \subseteq \zeta_i(U_i)$ with $\mathbb{A}_{r,1} := \{z;\, r < |z| < 1\}$ such that*

$$\frac{1}{K} \leq \left| \frac{\rho_{U_i}(z)}{\lambda_{\mathbb{D}}(z)} \right| \leq K$$

for all $z \in \mathbb{A}_{r,1}$.

We may now proceed with the proof of the Theorem.

Proof. (of Theorem 4.5). Choose collar charts (ζ_i, U_i) and annuli $\mathbb{A}_{r_i}$ satisfying the conclusion of Lemma 4.8 with constants K_i. Since there are only finitely many boundary curves we may assume that $K_i = K$ for some K for all i. Choose s_i such that $1 < s_i < r_i$ for each i, and let M be the subset of Σ given by

$$M = \Sigma \backslash \cup_{i=1}^{n} \{\zeta_i^{-1}(\mathbb{A}_{s_i})\}.$$

Clearly M is compactly contained in Σ.

Let $\alpha \in H_{-1,1}(\Sigma)$. Applying Lemma 4.7 we have that

$$\|\alpha\|_{\infty,M,\Sigma} \leq D_M \|\alpha\|_{2,\Sigma}$$

where D depends only on M. On the other hand, given α by definition there is a $\psi \in A_2^2(\Sigma)$ such that α locally has the expression $\rho_U(z)^{-2}\overline{\psi_U(z)}$, where ψ_U is the local expression for ψ. In particular α has the form $\rho_{U_i}(z)^{-2}\overline{\psi_{U_i}(z)}$ in ζ_i coordinates.

For each $i = 1, \ldots, n$, choose t_i such that $1 < t_i < s_i$. By Lemmas 4.6 and 4.8, we have the estimate

$$\begin{aligned}
\sup_{z \in \mathbb{A}_{r_i}} \rho_{U_i}(z)^{-2}|\psi_{U_i}(z)| &\leq K^2 \sup_{z \in \mathbb{A}_{r_i}} \lambda_{\mathbb{D}}(z)^{-2}|\psi_{U_i}(z)| \\
&\leq K^2 C(r_i, t_i) \left(\iint_{\mathbb{A}_{r_i}} \lambda_{\mathbb{D}}(z)^{-2}|\psi_{U_i}(z)|^2 \, dA \right)^{1/2} \\
&\leq K^3 C(r_i, t_i) \left(\iint_{\mathbb{A}_{r_i}} \rho_{U_i}(z)^{-2}|\psi_{U_i}(z)|^2 \, dA \right)^{1/2} \\
&\leq K^3 C(r_i, t_i) \|\alpha\|_{2,\Sigma}.
\end{aligned}$$

If we let $C = \max\{D_M, K^3C(r_1,t_1), \ldots, K^3C(r_n,t_n)\}$ we thus have proven that

$$\|\alpha\|_{\infty,\Sigma} \leq C\|\alpha\|_{2,\Sigma}.$$

□

Next, we observe an immediate but important corollary.

Corollary 4.9. *Let* Σ *be a bordered Riemann surface of type* (g,n).

$$\Omega_{-1,1}(\Sigma) \cap \mathrm{TBD}(\Sigma) = H_{-1,1}(\Sigma).$$

The remainder of this section is dedicated to showing that $H_{-1,1}(\Sigma)$ is complementary to the kernel of the restriction of the Bers embedding to $\mathrm{TBD}(\Sigma)$. In the next section, we will show that it is a model for the tangent space to WP-class Teichmüller space.

First, we will recapitulate some of the known facts regarding the kernel of the Bers embedding in the case of the standard Teichmüller space. One has the standard decomposition

$$L^{\infty}_{-1,1}(\mathbb{D}^*) = \mathcal{N}(\mathbb{D}^*) \oplus \Omega^{-1,1}(\mathbb{D}^*)$$

where

$$\mathcal{N}(\mathbb{D}^*) = \left\{ \mu \in L^{\infty}_{-1,1}(\mathbb{D}^*) \,:\, \iint_{\mathbb{D}^*} \mu\phi = 0 \;\; \forall \phi \in A_2^1(\mathbb{D}^*) \right\}$$

is the linear space of "infinitesimally trivial" Beltrami differentials, which are by classical results precisely the kernel of the derivative at the identity of the Bers embedding of the universal Teichmüller space [12, Chpt 3], [11, V.7]. We now define similar spaces on the bordered Riemann surface Σ of type (g, n).

$$\mathcal{N}(\Sigma) = \left\{ \mu \in L^\infty_{-1,1}(\Sigma) \,:\, \iint_\Sigma \mu\phi = 0 \quad \forall \phi \in A^1_2(\Sigma) \right\}.$$

The following theorem is standard, although often phrased in its equivalent form using Fuchsian groups.

Theorem 4.10. *Let Σ be a bordered Riemann surface of type (g, n). Then*

$$L^\infty_{-1,1}(\Sigma) = \mathcal{N}(\Sigma) \oplus \Omega_{-1,1}(\Sigma).$$

Furthermore $\mathcal{N}(\Sigma)$ is the kernel of the Bers embedding.

We sketch the proof in order to establish the notation and concepts for the proof of Theorem 4.11 ahead. Full details can be found in the references. We have (up to biholomorphism) that $\Sigma = \mathbb{D}^*/G$ for some Fuchsian group G. Define

$$L^\infty_{-1,1}(\mathbb{D}^*, G) = \left\{ \mu \in L^\infty_{-1,1}(\mathbb{D}^*) \,:\, \mu \circ g \frac{\overline{g'}}{g'} = \mu \quad \forall g \in G \right\}.$$

Also define

$$\mathcal{N}(\mathbb{D}^*, G) = \mathcal{N}(\mathbb{D}^*) \cap L^\infty_{-1,1}(\mathbb{D}^*, G)$$

and

$$\Omega_{-1,1}(\mathbb{D}^*, G) = \Omega_{-1,1}(\mathbb{D}^*) \cap L^\infty_{-1,1}(\mathbb{D}^*, G).$$

It is immediate that

$$L^\infty_{-1,1}(\mathbb{D}^*, G) = \mathcal{N}(\mathbb{D}^*, G) \oplus \Omega_{-1,1}(\mathbb{D}^*, G). \tag{4.12}$$

Clearly we may identify $L^\infty_{-1,1}(\mathbb{D}^*, G)$ with $L^\infty_{-1,1}(\Sigma)$ and $\Omega_{-1,1}(\mathbb{D}^*, G)$ with $\Omega_{-1,1}(\Sigma)$. We must show that $\mathcal{N}(\mathbb{D}^*, G)$ can be identified with $\mathcal{N}(\Sigma)$, and that $\mathcal{N}(\Sigma)$ is the kernel of the derivative of the Bers embedding at the point $[\Sigma, \mathrm{Id}, \Sigma]$. This latter fact is well known: let F be a fixed fundamental domain of the group G and temporarily let

$$A^1_2(F) = \left\{ \phi(z) \in L^2(\mathbb{D}^*) \,:\, \phi \text{ holo},\ \phi(g(z))g'(z)^2 dz^2 = \phi(z) \quad \forall g \in G, \text{ and } \iint_F |\phi(z)|\, dA < \infty \right\}$$

and temporarily let

$$N(G) = \left\{ \mu \in L^\infty_{-1,1}(\mathbb{D}^*, G) \,:\, \iint_F \mu\phi = 0\ \forall \phi \in A^1_2(F) \right\}.$$

It is clear that $N(G)$ can be identified with $\mathcal{N}(\Sigma)$. By [11, Chapter V, Theorem 7.2], $N(G)$ is the kernel of the derivative of the Bers embedding at the base point. It remains to show that $\mathcal{N}(\mathbb{D}^*, G)$ can be identified with $\mathcal{N}(\Sigma)$.

To do this we show that $\mathcal{N}(\mathbb{D}^*, G) = N(G)$. Let $\Theta : A^1_2(\mathbb{D}^*) \to A^1_2(F)$ be the Poincaré projection operator [11, V.7.3]. Let $\mu \in \mathcal{N}(\mathbb{D}^*) \cap L^\infty_{-1,1}(\mathbb{D}^*, G)$. Let $\phi \in A^1_2(F)$. Since Θ is surjective [11, Theorem V.7.1] there is a $\psi \in A^1_2(\mathbb{D}^*)$ such that $\Theta(\psi) = \phi$. By [11, Chapter V, (7.3)]

$$\iint_F \mu\phi = \iint_{\mathbb{D}^*} \mu\psi = 0,$$

so $\mu \in N(G)$.

Conversely assume that $\mu \in N(G)$. Let $\phi \in A^1_2(\mathbb{D}^*)$. By [11, Equation (7.3) V.7.3]

$$\iint_{\mathbb{D}^*} \mu\phi = \iint_F \mu(\Theta\phi) = 0.$$

So $\mu \in \mathcal{N}(\mathbb{D}^*) \cap L^\infty_{-1,1}(\mathbb{D}^*, G)$. Thus $\mathcal{N}(\mathbb{D}^*, G) = N(G)$.

We conclude that

$$L^\infty_{-1,1}(\Sigma) = \mathcal{N}(\Sigma) \oplus \Omega_{-1,1}(\Sigma). \tag{4.13}$$

It now follows that $H_{-1,1}(\Sigma)$ is complementary to the infinitesimally trivial differentials. Define

$$\mathcal{N}_r(\Sigma) = \mathcal{N}(\Sigma) \cap \mathrm{TBD}(\Sigma) = \mathcal{N}(\Sigma) \cap L^2_{-1,1}(\Sigma).$$

Thanks to Corollary 4.9 of Theorem 4.5, we have the following theorem.

Theorem 4.11. *Let Σ be a bordered Riemann surface of type (g, n).*

$$\mathrm{TBD}(\Sigma) = \mathcal{N}_r(\Sigma) \oplus H_{-1,1}(\Sigma).$$

Furthermore, $\mathcal{N}_r(\Sigma)$ is the intersection of the kernel of the Bers embedding at the base point with $L^2_{-1,1}(\Sigma)$; that is, it is the kernel of the restriction of the Bers embedding to $\mathrm{BD}(\Sigma)$.

Proof. The final statement follows from the above discussion. The decomposition follows from Theorem 4.10 by intersecting with $L^2_{-1,1}(\Sigma)$ and applying Corollary 4.9. □

We also require the following classical result. Let G be a Fuchsian group acting on $\mathbb{D}$, and let F be a fundamental domain for G. (We choose the cover $\mathbb{D}$ rather than $\mathbb{D}^*$ for the next few paragraphs in order to be consistent with the references and avoid minor convergence issues). For $(-1, 1)$ differentials ν-invariant under G, define the integral map

$$\mathbf{K}(\nu)(z) = \frac{3}{\pi}(1 - |z|^2)^2 \iint_{\mathbb{D}} \frac{1}{(1 - \bar{\zeta}z)^4} \overline{\nu(\zeta)}\, dA_\zeta\ . \tag{4.14}$$

We have not yet addressed convergence. We claim that

$$\mathbf{K} : L^\infty_{-1,1}(\mathbb{D}, G) \longrightarrow \Omega_{-1,1}(F) \tag{4.15}$$

and

$$\mathbf{K} : L^2_{-1,1}(\mathbb{D}, G) \longrightarrow H_{-1,1}(F) \tag{4.16}$$

are bounded, where $L^2_{-1,1}(\mathbb{D}, G)$ is the space of G-invariant $(-1, 1)$ differentials such that

$$\|\nu\|_{2,F} = \iint_F \frac{|\nu(z)|^2}{(1 - |z|^2)^2}\, dA < \infty.$$

This can be identified with $L^2_{-1,1}(\Sigma)$ if $\Sigma = \mathbb{D}/G$. This follows from [10, Lemma 3.4.9] and Proposition 4.1. Furthermore, the kernel of $\mathbf{K}|_{L^\infty_{-1,1}(\mathbb{D},G)}$ is just the infinitesimally trivial differentials $\mathcal{N}(F)$.

It is clear that these results can be written on the Riemann surface Σ rather than the fundamental domain. Restating the above results on Σ, and applying Theorem 4.11, we have the following theorem.

Theorem 4.12. *Let Σ be a bordered Riemann surface of type (g, n). There is a bounded projection $P : L^{\infty}_{-1,1}(\Sigma) \to \Omega_{-1,1}(\Sigma)$ such that the restriction*

$$P|_{\mathrm{BD}(\Sigma)} : \mathrm{BD}(\Sigma) \to H_{-1,1}(\Sigma)$$

is bounded with respect to the $L^2_{-1,1}(\Sigma)$ norm. The kernel of the restriction of P to $\mathrm{TBD}(\Sigma)$ is $\mathcal{N}_r(\Sigma)$.

Remark 4.13. In the last statement, we make use of the fact that the derivative of P is P itself. Note that P is linear on both spaces $L^2_{-1,1}(\Sigma)$ and $L^{\infty}_{-1,1}(\Sigma)$.

When combined with Theorem 3.25, we get the following crucial consequence.

Theorem 4.14. *Let Σ be a bordered Riemann surface of type (g, n) such that $2g-2+n > 0$. Assume that $\mathbf{v}$ is a tangent vector to $T_{\mathrm{WP}}(\Sigma)$ at $[\Sigma, Id, \Sigma]$. There is a holomorphic curve $t \mapsto [\Sigma, f_t, \Sigma_t]$, $|t| < \delta$, in $T_{\mathrm{WP}}(\Sigma)$ through $[\Sigma, Id, \Sigma]$ at $t = 0$ such that the Beltrami differential μ_t of f_t is in $H_{-1,1}(\Sigma)$ for all $|t| < \delta$, μ_t is holomorphic in t, and such that the tangent vector to this curve at $[\Sigma, Id, \Sigma]$ is $\mathbf{v}$.*

Proof. Let $\mathbf{v}$ be a tangent vector to $T_{\mathrm{WP}}(\Sigma)$ at $[\Sigma, \mathrm{Id}, \Sigma]$. Let $[\Sigma, f_t, \Sigma_t]$ be a holomorphic curve through $[\Sigma, \mathrm{Id}, \Sigma]$ at $t = 0$. By Theorem 3.25 we can assume that the Beltrami differential of f_t is in $\mathrm{BD}(\Sigma)$ for $|t| < \delta$ for some $\delta > 0$. By Theorem 4.12 if we set $\mu_t = P(\mu(f_t))$ the resulting Beltrami differential is in $H_{-1,1}(\Sigma)$ for all $|t| < \delta$. Furthermore solving the Beltrami equation to obtain $[\Sigma, g_t, \Sigma_t]$, the tangent vector to $[\Sigma, g_t, \Sigma_t]$ at $t = 0$ must be the same as $[\Sigma, f_t, \Sigma_t]$ by Theorem 4.11, since infinitesimally trivial differentials are in the kernel of the Bers embedding. ☐

4.2. $H_{-1,1}(\Sigma)$ **model of Weil-Petersson class Teichmüller space.** In the previous section we showed that the tangent vector at the identity of every differentiable curve through the base element of the WP-class Teichmüller space is in $H_{-1,1}(\Sigma)$. In this section, we show that the WP-class Teichmüller space is locally biholomorphic to $H_{-1,1}(\Sigma)$. This is the main result of the paper and it allows us to define a convergent WP-metric in Section 4.3. This also gives an alternate description of the complex structure.

We proceed as follows. First, in Theorem 4.15 below, we show that the restriction to $H_{-1,1}(\Sigma)$ of the map Φ into $T_{\mathrm{WP}}(\Sigma)$ taking Beltrami differentials to the solution of the Beltrami equation is holomorphic on some open neighborhood of 0. This result uses the Schiffer variation coordinates of Section 3.2 along with Theorem 4.5 and Lemma 4.8. Once this is established, we apply the preparation Theorem 3.25 together with the inverse function theorem to establish that in fact Φ is a biholomorphism on some open ball. In [18] we showed that change of base point is a biholomorphism. Using this fact allows us to show in Theorem 4.17 that any point has a neighbourhood biholomorphic to a ball in $H_{-1,1}(\Sigma)$.

Denoting the unit ball of $\Omega_{-1,1}(\Sigma)$ by $\Omega_{-1,1}(\Sigma)_1$, define the map

$$\begin{aligned} \check{\Phi} : \Omega_{-1,1}(\Sigma)_1 &\longrightarrow T(\Sigma) \\ \mu &\longmapsto [\Sigma, f_\mu, \Sigma_1]. \end{aligned}$$

where $f_\mu : \Sigma \to \Sigma_1$ is a solution to the Beltrami equation with differential μ. Let

$$\Phi : H_{-1,1}(\Sigma) \longrightarrow T_{\mathrm{WP}}(\Sigma)$$

be the restriction of $\check{\Phi}$ to $H_{-1,1}(\Sigma)$. Note that since $H_{-1,1}(\Sigma) \subseteq L^2_{-1,1}(\Sigma)$, by Definition 3.3 Φ maps into $T_{\mathrm{WP}}(\Sigma)$. We will keep the distinction between Φ and $\check{\Phi}$, even though Φ is the restriction of $\check{\Phi}$, in order to indicate the change in norm on the domain.

Theorem 4.15. *Let Σ be a bordered Riemann surface of type (g,n) such that $2g-2+n>0$. Then there is an open neighbourhood B of 0 in $H_{-1,1}(\Sigma)$, such that the map Φ is holomorphic on B.*

Proof. Fix a $\tau \in \mathrm{QS}_{\mathrm{WP}}(\Sigma)$ (which exists by Theorem 3.12) and sew caps on Σ via τ to obtain a punctured Riemann surface Σ^P. We assume moreover that τ is an analytic parametrization, so that $\tilde{\tau}_i$ has an analytic extension to an open neighbourhood of $\overline{\mathbb{D}}$ for each $i = 1, \ldots, n$. Since the complex structure is independent of τ [18, Corollary 4.22], there is no loss of generality in this assumption.

Let $\hat{B}$ be the open unit ball in $\Omega_{-1,1}(\Sigma)$ centred at 0. Since inclusion $\iota : H_{-1,1}(\Sigma) \to \Omega_{-1,1}(\Sigma)$ is holomorphic by Theorem 4.5, we see that $B = \iota^{-1}(\hat{B})$ is open in $H_{-1,1}(\Sigma)$ and contains 0. Given a $\mu \in B$, we let $\hat{\mu}$ be the Beltrami differential obtained from μ by setting it to be zero on the caps. Define a map into the Teichmüller space $T(\Sigma^P)$ by

$$\begin{aligned} \Xi : B &\longrightarrow T(\Sigma^P) \\ \mu &\longmapsto [\Sigma^P, f_{\hat{\mu}}, \Sigma^P_{\hat{\mu}}] \end{aligned}$$

where $\Sigma^P_{\hat{\mu}}$ and $f_{\hat{\mu}}$ are defined as in expressions (5.1) and (5.2). In particular, $f_{\hat{\mu}} : \Sigma^P \to \Sigma^P_{\hat{\mu}}$ is a solution to the Beltrami equation on Σ^P with dilatation $\hat{\mu}$. We claim that Ξ is holomorphic. This is because it can be written as the composition of four holomorphic maps. That is, $\Xi = \Psi \circ \mathrm{ext} \circ i \circ \iota$ where (1) inclusion $\iota : H_{-1,1}(\Sigma) \to \Omega_{-1,1}(\Sigma)$ is holomorphic by Theorem 4.5; (2) the inclusion $i : \Omega_{-1,1}(\Sigma) \to L^\infty_{-1,1}(\Sigma)$ is obviously holomorphic; (3) $\mathrm{ext} : L^\infty_{-1,1}(\Sigma) \mapsto L^\infty_{-1,1}(\Sigma^P)$ is holomorphic since by direct computation it is Gâteaux holomorphic and locally (in fact, globally) bounded; finally (4) the solution to the Beltrami equation $\Psi : L^\infty_{-1,1}(\Sigma^P) \to T(\Sigma^P)$ is holomorphic (see [12]). Observe that $\Xi = \mathcal{F} \circ \Pi \circ \Phi$; however note that this factorization was not necessary in the forgoing proof of holomorphicity of Ξ .

A word on the proof may be helpful. In order to write Φ in coordinates $\mathcal{G} \circ \Phi$, we must write points in the image of $\Pi \circ \Phi$ as elements of $F(U,S,\Omega)$. However, a given point in the image of $\Pi \circ \Phi$ will not be of the Schiffer variation form $(\Sigma^P, \nu_\epsilon, \Sigma_\epsilon, \psi)$. In order to reach this form, we need to compose by some biholomorphism σ_ϵ of Σ^P_ϵ and invoke the Teichmüller equivalence under homotopy. This explains the presence of some extra compositions. Furthermore, as mentioned earlier, in order to make rigorous statements regarding holomorphicity, we must treat the Schiffer variation as a marked holomorphic family. The reader should bear these two points in mind in what follows.

Fix an n-chart (ζ, E) on Σ^P_1 such that $\overline{\tilde{\tau}_i(\mathbb{D})} \subseteq E_i$ for each $i = 1, \ldots, n$. Choose an open set $K_i \subset \zeta_i(E_i)$ containing $\zeta_i(\overline{\tilde{\tau}_i(\mathbb{D})})$ for $i = 1, \ldots, n$ such that $\overline{K_i}$ is compactly contained in $\zeta_i(E_i)$. Let $U_i \subset \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ be open sets chosen so that $\overline{\phi_i(\mathbb{D})} \subseteq K_i$ for all ϕ_i in U_i, $i = 1, \ldots, n$. This is possible by [17, Theorem 3.4]. Let $\mathfrak{S} : \Omega \to T(\Sigma^P)$ be a Schiffer variation based at $[\Sigma, Id, \Sigma]$ which is compatible with the n-chart, and let $F(U,S,\Omega)$ be the corresponding open set in $\widetilde{T}_{\mathrm{WP}}(\Sigma^P)$. Let $\pi : S(\Omega, D) \to \Omega$ be the marked Schiffer family corresponding to

$\mathfrak{S}$, with strong global trivialization

$$\begin{aligned} \theta : \Omega \times \Sigma^P &\to S(\Omega, D) \\ (\epsilon, q) &\mapsto (\epsilon, \nu_\epsilon(q)), \end{aligned}$$

as defined in (3.3).

By Theorem 3.16 there is a biholomorphic map

$$\begin{aligned} \Gamma : S(\Omega, D) &\longrightarrow \pi_T^{-1}(\mathfrak{S}(\Omega)) \\ (\epsilon, p) &\longmapsto \big([\Sigma^P, \nu_\epsilon, \Sigma_\epsilon], \sigma_\epsilon(p)\big) \end{aligned}$$

where $\sigma_\epsilon : \Sigma^P_{1,\epsilon} \to \Sigma_{\hat{\mu}} = \pi_T^{-1}([\Sigma^P, \nu_\epsilon, \Sigma_\epsilon])$ is a biholomorphism depending holomorphically on ϵ, and $f_{\hat{\mu}}^{-1} \circ \sigma_\epsilon \circ \nu_{\epsilon(\mu)}$ is homotopic to the identity (see Remark 5.6). Thus $[\Sigma^P, \nu_{\epsilon(\mu)}, \Sigma_\epsilon] = [\Sigma^P, \sigma_\epsilon \circ \nu_{\epsilon(\mu)}, \Sigma^P_{\hat{\mu}}] = [\Sigma^P, f_{\hat{\mu}}, \Sigma^P_{\hat{\mu}}]$. By restricting B sufficiently (namely, to $\Xi^{-1}(\mathfrak{S}(\Omega))$ - note that by holomorphicity of Ξ the inverse image of $\mathfrak{S}(\Omega)$ is open), one obtains

$$\begin{aligned} \theta^{-1} \circ \Gamma^{-1} : \pi_T^{-1}(\Xi(B)) &\longrightarrow \Omega \times \Sigma^P \\ \big([\Sigma^P, f_{\hat{\mu}}, \Sigma^P_{\hat{\mu}}], w\big) &\longmapsto \Big(\epsilon(\mu), \nu^{-1}_{\epsilon(\mu)} \circ \sigma^{-1}_{\epsilon(\mu)}(w)\Big) \end{aligned} \tag{4.17}$$

where $\Gamma \circ \theta$ is a strong global trivialization. In particular $\epsilon(\mu)$ is holomorphic in μ (as a function on $H_{-1,1}(\Sigma)$).

By Theorem 3.22 it suffices to show that

$$\begin{aligned} \Pi \circ \Phi : B &\longrightarrow \widetilde{T}_0(\Sigma^P) \\ \mu &\longmapsto \big[[\Sigma^P, f_{\hat{\mu}}, \Sigma^P_{\hat{\mu}}], f_{\hat{\mu}} \circ \tilde{\tau}_1, \ldots, f_{\hat{\mu}} \circ \tilde{\tau}_n\big] \end{aligned}$$

is a biholomorphism. Since $\sigma^{-1}_{\epsilon(\mu)} \circ f_{\hat{\mu}}$ is homotopic to $\nu_{\epsilon(\mu)}$ we have by the equivalence relation of Definition 3.7

$$\big[[\Sigma^P, f_{\hat{\mu}}, \Sigma^P_{\hat{\mu}}], f_{\hat{\mu}} \circ \tilde{\tau}_1, \ldots, f_{\hat{\mu}} \circ \tilde{\tau}_n\big] = \big[[\Sigma^P, \nu_\epsilon, \Sigma_\epsilon], \sigma_\epsilon^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_1, \ldots, \sigma_\epsilon^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_n\big]\,.$$

Therefore, in the coordinates $\mathcal{G}$ defined in (3.5), $\Pi \circ \Phi$ is written

$$\begin{aligned} \mathcal{G} \circ \Pi \circ \Phi : W &\longrightarrow \mathbb{C}^d \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \\ \mu &\longmapsto \Big(\epsilon(\mu), \Big(\zeta_1 \circ \nu^{-1}_{\epsilon(\mu)} \circ \sigma^{-1}_{\epsilon(\mu)} \circ f_{\hat{\mu}} \circ \tilde{\tau}_1, \ldots, \zeta_n \circ \nu^{-1}_{\epsilon(\mu)} \circ \sigma^{-1}_{\epsilon(\mu)} \circ f_{\hat{\mu}} \circ \tilde{\tau}_n\Big)\Big)\,, \end{aligned}$$

where d is the dimension of $T(\Sigma^P)$. It was shown above that the first component is holomorphic in μ. Thus we only need to show that the second component is holomorphic in μ, as a map into $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

To this end, it is enough to show that the second component is Gâteaux holomorphic and locally bounded (see e.g. [6]). Thus we fix $\omega \in B$ and consider the maps

$$t \longmapsto \zeta_i \circ \nu^{-1}_{\epsilon(\omega + t\mu)} \circ \sigma^{-1}_{\epsilon(\omega+t\mu)} \circ f_{\widehat{\omega+t\mu}} \circ \tilde{\tau}_i. \tag{4.18}$$

where t is restricted to some open neighbourhood of $0 \in \mathbb{C}$ so that $\omega + t\mu \in B$. Since Γ^{-1} is holomorphic, and θ^{-1} is holomorphic in ϵ for fixed $z \in E_i$, we can conclude that $\nu^{-1}_{\epsilon(\omega+t\mu)} \circ \sigma^{-1}_{\epsilon(\omega+t\mu)}$ depends holomorphically on t. Furthermore $f_{\hat{\mu}}$ depends holomorphically on $\hat{\mu}$ (because it is a strong local trivialization for the Teichmüller curve as defined in (5.2)), and hence $f_{\widehat{\omega+t\mu}}$ depends holomorphically on t for fixed z. Since

$$\zeta_i \circ \nu^{-1}_{\epsilon(\omega+t\mu)} \circ \sigma^{-1}_{\epsilon(\omega+t\mu)} \circ f_{\widehat{\omega+t\mu}} \circ \tilde{\tau}_i$$

is also holomorphic in z on $\mathbb{D}$, by Hartogs' theorem it is jointly holomorphic, and thus all z derivatives are holomorphic in t. So

$$t \mapsto (\zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ f_{\widehat{\omega+t\mu}} \circ \tilde{\tau}_i)'(0)$$

is holomorphic in t. We now need to show that

$$t \mapsto \mathcal{A} \circ \zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ f_{\widehat{\omega+t\mu}}$$

is holomorphic in t (as a map into $A_2^1(\mathbb{D})$) where

$$\mathcal{A}(h) = \frac{h''(z)}{h'(z)}.$$

Let $e_z : \mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}} \to \mathbb{C}$ denote point evaluation at $z \in E_i$. Since the point evaluations are a separating set of continuous linear functionals, to show that $\mathcal{G}$ is Gâteaux holomorphic it is enough (see [6]) to prove that

$$e_z \circ \mathcal{A} \circ \zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ f_{\widehat{\omega+t\mu}}$$

is holomorphic in t for each z, and is locally bounded. The holomorphicity in t for fixed z follows from the same argument as above.

Local boundedness will follow from the local boundedness of $\mathcal{G} \circ \Pi \circ \Phi$. Thus the local boundedness of $\mathcal{G} \circ \Pi \circ \Phi$ is the only remaining step. Recall that by construction, for every μ in B, $\Pi \circ \Phi(\mu) \in F(U, S, \Omega)$; in particular, the closure of $\zeta_i \circ \nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1}(f_{\hat{\mu}} \circ \tilde{\tau}_i(\mathbb{D}))$ is in K_i. Thus, since θ is continuous, if we further restrict B so that $\epsilon(\mu)$ is a subset of a compact set $\Omega' \subseteq \Omega$ containing 0, we can guarantee that $\Gamma^{-1} \circ \pi_T^{-1} \Xi(B)$ is contained in the compact set

$$\{(\epsilon, \nu_\epsilon(\overline{K_i})) \,:\, \epsilon \in \Omega'\} \subseteq S(\Omega, D).$$

This takes care of the local boundedness of $\epsilon(\mu)$.

Fix an analytic extension of $\tilde{\tau}$ to a disk $D_s = \{z \,:\, |z| < s\}$ for some $s > 1$. Since $f_{\hat{\mu}}(z)$ is a continuous function of both z and μ (again, because it is a strong local trivialization for the Teichmüller curve), and the same thing also holds for $\nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1}$, there is a uniform $R > 1$ such that for every $\mu \in B$ in this ball, we have that

$$\nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_i(D_R) \subseteq E_i$$

where E_i is the domain of the n-chart (ζ_i, E_i) and $D_R = \{z \,:\, |z| < R\}$. Fix $1 < r < R$ and let $F_{\hat{\mu}}$ be any quasiconformal extension of $\zeta_i \circ \nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_i$ to $\mathbb{C}$ agreeing with the original map on $\{z \,:\, |z| \leq r\}$. This quasiconformal map represents the same element of $\mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}}$. The L^2 norm of the extension satisfies

$$\iint_{\mathbb{D}^*} \left| \frac{\overline{\partial} F_{\hat{\mu}}(z)}{\partial F_{\hat{\mu}}(z)} \right|^2 \frac{1}{(1-|z|^2)^2} \, dA \leq C\|\mu\|_{2,\Sigma} + \text{ Hyperbolic Area } (\{z \,:\, |z| > r\} \cup \{\infty\}) \tag{4.19}$$

where the first term is a bound on the dilatation in $|z| < r$ arising from Lemma 4.8, and the second term bounds the dilatation on $|z| > r$ using only the fact that the complex dilatation of $\tilde{f}_{\hat{\mu}}$ is less than one. It is clear that both terms on the right hand side are finite and bounded by a fixed constant.

Referring to (2.2) we see that we need to show that $|F'_{\hat{\mu}}(0)|$ and

$$\iint_{\mathbb{D}} \left| \frac{F''_{\hat{\mu}}(z)}{F'_{\hat{\mu}}(z)} \right|^2 \, dA$$

are bounded. Since $\overline{F_{\hat{\mu}}(\mathbb{D})}$ is a subset of K_i and in particular bounded in some disk, by the Schwarz lemma $|F'_{\hat{\mu}}(0)| \leq M$ for some fixed $M > 0$. By Lemma 3.23, we have for some $\delta > 0$ and $|t| < \delta$,

$$\iint_{\mathbb{D}} \left| \frac{F''_{\hat{\mu}}(z)}{F'_{\hat{\mu}}(z)} \right|^2 dA \approx \iint_{\mathbb{D}} |\mathcal{S}F_{\hat{\mu}}(z)|^2 (1-|z|^2)^2 \, dA + \left| \frac{F''_{\hat{\mu}}(0)}{F'_{\hat{\mu}}(0)} \right|^2$$

where S denotes the Schwarzian derivative. By the classical second Taylor coefficient estimate for a univalent map of $\mathbb{D}$ the second term on the right hand side is bounded by $4|F'_{\hat{\mu}}(0)| \leq 4M$. Finally, by [7, Theorem 2] we have the estimate

$$\iint_{\mathbb{D}} |\mathcal{S}F_{\hat{\mu}}(z)|^2 (1-|z|^2)^2 \, dA \leq C \iint_{\mathbb{D}^*} \frac{|\hat{\mu}(z)|^2}{(1-|z|^2)^2} \, dA.$$

(Note that in [7, Theorem 2] the roles of $\mathbb{D}$ and $\mathbb{D}^*$ are reversed; however the left and right side are invariant under reflection in the circle $z \mapsto 1/z$ so it immediately implies the estimate above). Hence by (4.19) Φ is locally bounded. This concludes the proof of the theorem. $\square$

Theorem 4.16. *Let Σ be a bordered Riemann surface of type (g,n) such that $2g-2+n>0$. There is an open neighbourhood U of $0 \in H_{-1,1}(\Sigma)$ on which Φ is a biholomorphism.*

Proof. We will show first that $D\Phi(0)$ is a topological isomorphism. Since we have already shown that Φ is holomorphic, it follows that $D\Phi(0)$ is bounded, and thus by the open mapping theorem it suffices to show that $D\Phi(0)$ is bijective.

We first show that $D\Phi(0)$ is surjective. Let $\mathbf{v}$ be a tangent vector to $T_{\mathrm{WP}}(\Sigma)$ at $[\Sigma, \mathrm{Id}, \Sigma]$. By Theorem 4.14 we may find a holomorphic curve $\alpha(t) = [\Sigma, g_t, \Sigma_t]$ such that the Beltrami differential μ_t of g_t is in $H_{-1,1}(\Sigma)$, has tangent vector $\mathbf{v}$ at 0 and is holomorphic in t. In that case $\alpha(t) = \Phi(\mu_t)$ and so

$$\mathbf{v} = \frac{d}{dt}\, \alpha\Big|_{t=0} = D\Phi\left(\left. \frac{d\mu_t}{dt} \right|_{t=0} \right)$$

which proves the claim.

Next we show that $D\Phi(0)$ is injective. The kernel is trivial since

$$\ker D\Phi(0) = \ker D\check{\Phi}(0) \cap H_{-1,1}(\Sigma) = \{0\}$$

since $\ker D\check{\Phi}(0)$ is trivial by Theorem 4.10.

Since Φ is holomorphic by Theorem 4.15, by the inverse function theorem [9] Φ has a C^1 inverse in a neighbouhood of 0. Thus there is a U on which Φ is a biholomorphism. $\square$

Finally, we show that $T_{\mathrm{WP}}(\Sigma)$ possesses an atlas of charts modelled on $H_{-1,1}(\Sigma)$. We require a change of base point. Let $[\Sigma, f, \Sigma_0] \in T_{\mathrm{WP}}(\Sigma)$. Define the change of base point map by

$$\begin{aligned} f^* : T_{\mathrm{WP}}(\Sigma_0) &\longrightarrow T_{\mathrm{WP}}(\Sigma) \\ [\Sigma_0, \rho, \Sigma_1] &\longmapsto [\Sigma, \rho \circ f, \Sigma_1]. \end{aligned}$$

It was shown in [18, Section 4.3] that this map is a biholomorphism. Thus we may conclude that

Theorem 4.17. *Let Σ be a bordered Riemann surface of type (g,n) $2g-2+n>0$. For each point $[\Sigma, f, \Sigma_0]$, there is an open neighbourhood B of 0 in $H_{-1,1}(\Sigma)$ such that*

$$\begin{aligned} \Phi_{(\Sigma,f,\Sigma_0)} : B &\longrightarrow T_{\mathrm{WP}}(\Sigma) \\ \mu &\longmapsto [\Sigma, f_\mu \circ f, f_\mu(\Sigma_0)] \end{aligned}$$

is a biholomorphism onto an open neighbourhood of $[\Sigma, f, \Sigma_0]$. In particular, the collection of such biholomorphisms forms an atlas on $T_{\mathrm{WP}}(\Sigma)$.

Proof. This follows immediately from Theorem 4.16 and the fact that f^* is a biholomorphism. □

Note that the map f^* is independent of the choice of representative of $[\Sigma, f, \Sigma_0]$.

4.3. **The explicit Weil-Petersson metric.** We are now ready to define the Weil-Petersson metric on the tangent space at the identity, which is done as follows. Any pair of tangent vectors can be represented by a pair of Beltrami differentials $\mu, \nu \in H_{-1,1}(\Sigma)$. Let $\mu, \nu \in H_{-1,1}(\Sigma)$ be two representatives of elements of the tangent space at the identity of the refined Teichmüller space of Σ^B. For coordinates on an open set U containing the set W we define the local integral as in Section 2.1. Assuming that $\mu = \mu_U d\bar{z}/dz$ and $\nu = \nu_U d\bar{z}/dz$ in local coordinates, if W is a measurable set contained in U we can define the integral

$$\iint_W \mu_U(z)\overline{\nu}_U(z)\rho_U(z)^2\, dA$$

where $\rho_U(z)^2|dz|^2$ is the local expression for the hyperbolic metric on Σ in the chart U. It is easily checked that this expression is invariant under change of coordinates. Using a partition of unity subordinate to a cover we can extend this to an integral over Σ, which we will denote by

$$\langle \mu, \nu \rangle_{[\Sigma,\mathrm{Id},\Sigma]} = \iint_\Sigma \mu\overline{\nu} dA_\Sigma \tag{4.20}$$

where dA_Σ is the hyperbolic area measure on Σ.

One may also represent tangent space elements as lying in the space of quadratic differentials $A_2^2(\Sigma)$; that is for quadratic differentials $\alpha, \beta \in A_2^2(\Sigma)$ given by

$$\alpha = \mathfrak{B}^{-1}(\mu) \quad \beta = \mathfrak{B}^{-1}(\nu)$$

we can define the integral

$$(\alpha, \beta)_{[\Sigma,\mathrm{Id},\Sigma]} = \langle \mathfrak{B}(\alpha), \mathfrak{B}(\beta) \rangle_{[\Sigma,\mathrm{Id},\Sigma]}\,.$$

Finally, we observe that by changing the base point using f^*, we may define the Weil-Petersson metric at arbitrary points as follows. For a change of base point map f^* denote its derivative by Df^*.

Definition 4.18. Let Σ be a bordered Riemann surface of type (g,n) and let $[\Sigma, f, \Sigma_0] \in T_{\mathrm{WP}}(\Sigma)$. For $\mathbf{v}, \mathbf{w} \in T_{[\Sigma,f,\Sigma_0]}T_{\mathrm{WP}}(\Sigma)$ define the generalized Weil-Petersson metric by

$$\langle \mathbf{v}, \mathbf{w} \rangle_{[\Sigma,f,\Sigma_0]} = \left\langle D(f^{-1})^*\mathbf{v}, D(f^{-1})^*\mathbf{w} \right\rangle_{[\Sigma_0,\mathrm{Id},\Sigma_0]}\,.$$

Since f^* is independent of the representative $[\Sigma, f, \Sigma_0]$, this is well-defined.

One may define a similar expression in terms of the quadratic differentials.

Remark 4.19. There is no transitive group action on $T_{\mathrm{WP}}(\Sigma)$ to make use of, and we cannot lift the picture to the cover. Thus we cannot hope to say what "right invariance" even means, as we can in the case of the universal Teichmüller space. This is the unique metric on $T_{\mathrm{WP}}(\Sigma)$ which is invariant under change of base point and agrees with (4.20) for a single choice of base point.

5. Appendix: marked holomorphic families

In this appendix we give a very brief account of the aspects of the theory of marked holomorphic families and the universality of the Teichmüller curve, which we needed in Section 3.2. These results are due to Earle and Fowler [3], and we include this account only for the convenience of the reader. A full treatment appears in [3], and also in the books [8, 12].

Definition 5.1. A holomorphic family of complex manifolds is a pair of connected complex manifolds (E, B) together with a surjective holomorphic map $\pi : E \to B$ such that (1) π is topologically a locally trivial fiber bundle, and (2) π is a split submersion (that is, the derivative is a surjective map whose kernel is a direct summand).

Definition 5.2. A *morphism of holomorphic families* from (E', B') and (E, B) is a pair of holomorphic maps (α, β) with $\alpha : B' \to B$ and $\beta : E' \to E$ such that

$$\begin{array}{ccc} E' & \xrightarrow{\beta} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\alpha} & B \end{array}$$

commutes, and for each fixed $t \in B'$, the restriction of β to the fiber $\pi'^{-1}(t)$ is a biholomorphism onto $\pi^{-1}(\alpha(t))$.

Throughout, (E, B) will be a holomorphic family of Riemann surfaces; that is, each fiber $\pi^{-1}(t)$ is a Riemann surface.

Let Σ be a punctured Riemann surface of type (g, n). This fixed surface Σ will serve as a model of the fiber. Let U be an open subset of B.

Definition 5.3.

(1) A *local trivialization* of $\pi^{-1}(U)$ is a homeomorphism $\theta : U \times \Sigma \to E$ such that $\pi(\theta(t, x)) = t$ for all $(t, x) \in U \times \Sigma$.

(2) A local trivialization θ is a *strong local trivialization* if for fixed $x \in \Sigma$, $t \mapsto \theta(t, x)$ is holomorphic, and for each $t \in U$, $x \mapsto \theta(t, x)$ is a quasiconformal map from Σ onto $\pi^{-1}(t)$.

(3) $\theta : U \times \Sigma \to E$ and $\theta' : U \times \Sigma \to E$ are *compatible* if and only if $\theta'(t, x) = \theta(t, \phi(t, x))$ where for each fixed t, $\phi(t, x) : \Sigma \to \Sigma$ is a quasiconformal homeomorphism that is homotopic to the identity rel boundary.

(4) A *marking* $\mathcal{M}$ for $\pi : E \to B$ is a set of equivalence classes of compatible strong local trivializations that cover B.

(5) A *marked holomorphic family of Riemann surfaces* is a holomorphic family of Riemann surfaces with a specified marking.

Remark 5.4. Let θ and θ' be compatible strong local trivializations. For each fixed $t \in U$, $[\Sigma, \theta(t,\cdot), \pi^{-1}(t)] = [\Sigma, \theta'(t,\cdot), \pi^{-1}(t)]$ in $T(\Sigma)$, so a marking specifies a Teichmüller equivalence class for each t.

We now define the equivalence of marked families.

Definition 5.5. A *morphism of marked holomorphic families* from $\pi' : E' \to B'$ to $\pi : E \to B$ is a pair of holomorphic maps (α, β) with $\beta : E' \to E$ and $\alpha : B' \to B$ such that

(1) (α, β) is a morphism of holomorphic families, and
(2) the markings $B' \times \Sigma \to E$ given by $\beta(\theta'(t,x))$ and $\theta(\alpha(t), x)$ are compatible.

The second condition says that (α, β) preserves the marking.

Remark 5.6 (relation to Teichmüller equivalence). Define $E = \{(s, Y_s)\}_{s\in B}$ and $E' = \{(t, X_t)\}_{t\in B'}$ to be marked families of Riemann surfaces with markings $\theta(s,x) = (s, g_s(x))$ and $\theta'(t,x) = (t, f_t(x))$ respectively. Assume that (α, β) is a morphism of marked families, and define σ_t by $\beta(t,y) = (\alpha(t), \sigma_t(y))$. Then $\beta(\theta'(t,x)) = (\alpha(t), \sigma_t(f_t(x)))$ and $\theta(\alpha(t), x) = (\alpha(t), g_{\alpha(t)}(x))$. The condition that (α, β) is a morphism of marked families is simply that $\sigma_t \circ f_t$ is homotopic rel boundary to $g_{\alpha(t)}$. That is, when $s = \alpha(t)$, $[\Sigma, f_t, X_t] = [\Sigma, g_s, Y_s]$ via the biholomorphism $\sigma_t : X_t \to Y_s$.

Let $\mathbb{H}$ be the upper-half plane and G be a Fuchsian group such that $\Sigma = \mathbb{H}/G$ is a punctured Riemann surface (thus $2g - 2 + n > 0$). Let $T(G)$ be the "μ-model" of the Teichmüller space of Σ. Let μ be a Beltrami differential on Σ and let $\tilde{\mu}$ be the lift of μ to $\mathbb{H}$ and extended by 0 to the lower half plane. Let w^μ be the normalized solution of the Beltrami equation on $\mathbb{C}$ with dilatation $\tilde{\mu}$. The *Bers fiber space* is the subset $F(G) \subset T(G) \times \mathbb{C}$ given by

$$F(G) = \{([\mu], z) \mid [\mu] \in T(G), z \in w^\mu(\mathbb{H})\}.$$

Let $\mathcal{T}(G) = F(G)/G$. The group G acts freely and properly discontinuously by biholomorphisms on $F(G)$ and the quotient is a marked holomorphic family. We will define the marking below.

Definition 5.7. The marked holomorphic family of Riemann surfaces

$$\begin{aligned} \pi_T : \mathcal{T}(G) &\longrightarrow T(G) \\ ([\mu], z) &\longmapsto [\mu] \end{aligned}$$

is called the *Teichmüller curve*.

Let $G^\mu = w^\mu \circ G \circ (w^\mu)^{-1}$. The fiber above a point $[\mu] \in T(G)$ is the canonical Riemann surface

$$\Sigma_\mu = w^\mu(\mathbb{H})/G^\mu \tag{5.1}$$

which is independent of the Teichmüller equivalence class representative. The map w^μ uniquely defines a map

$$f_\mu : \Sigma \to \Sigma_\mu. \tag{5.2}$$

Note that while the boundary values of these maps are independent of the Teichmüller equivalence class representative, the maps themselves are not.

Let $L^{\infty}_{-1,1}(\Sigma)_1$ denote the open unit ball in $L^{\infty}_{-1,1}(\Sigma)$ and let U be an open subset of $T(\Sigma)$ for which a holomorphic section of the fundamental projection $L^{\infty}_{-1,1}(\Sigma)_1 \to T(\Sigma)$ exists. The strong local trivialization

$$\theta : U \to \pi_T^{-1}(U)$$

is defined by $\theta([\mu], z) = f_\mu(z)$

The following universal property of $\mathcal{T}(\Sigma)$ (see [3, 8, 12]) is all that we need for our purposes.

Theorem 5.8 (Universality of the Teichmüller curve)**.** *Let $\pi : E \to B$ be a marked holomorphic family of Riemann surfaces with fibre model Σ of type (g, n) with $2g - 2 + n > 0$, and trivialization θ. Then there exists a unique morphism (α, β) of marked families from $\pi : E \to B$ to $\pi_T : \mathcal{T}(\Sigma) \to T(\Sigma)$. Moreover, the canonical "classifying" map $\alpha : B \to T(\Sigma)$ is given by $\alpha(t) = [\Sigma, \theta(t, \cdot), \pi^{-1}(t)]$.*

References

[1] Ahlfors, L. V. and Sario L. *Riemann Surfaces*, Princeton University Press, 1960.

[2] G. Cui, *Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces*, Sci. China Ser. A **43** (2000), no. 3, 267–279.

[3] Earle, C. J. and Fowler, R. S. *Holomorphic families of open Riemann surfaces*, Math. Ann. **270** (1985), no. 2, 249–273.

[4] Friedan, D. and Shenker, S. *The analytic geometry of two-dimensional conformal field theory*, Nuclear Physics B **281** (1987), 509–545.

[5] Gardiner, F. P. *Schiffer's interior variation and quasiconformal mapping*, Duke Math. J. **42** (1975), 371–380.

[6] Grosse-Erdmann, K.-G. *A weak criterion for vector-valued holomorphy*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 2, 399–411.

[7] Hui, G. *Integrable Teichmüller spaces*, Sci. China Ser. A **43** (2000), no 1 , 47–58.

[8] Hubbard, J. H., with contributions by Douady, A., Dunbar, W., Roeder, R., Bonnot, S., Brown, D. Hatcher, A. Hruska, C. and Mitra, S. *Teichmüller theory and applications to geometry, topology and dynamics. Volume* 1 : *Teichmüller theory.* Matrix Editions, Ithaca, NY, 2006.

[9] Lang, S. *Differential and Riemannian manifolds*, Third edition. Graduate Texts in Mathematics, 160. Springer-Verlag, New York, 1995.

[10] Lehner, *Automorphic forms*. In Harvey, W. J. *Discrete groups and automorphic functions*, (Proc. Conf., Cambridge, 1975), pp. 73–120. Academic Press, London, 1977.

[11] Lehto, O. *Univalent functions and Teichmller spaces*, Graduate Texts in Mathematics, 109. Springer-Verlag, New York, 1987.

[12] Nag, S. *The Complex Analytic Theory of Teichmüller Spaces*, Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1988.

[13] Nag, S. and Verjovsky, A. Diff($\mathbb{S}^1$) *and the Teichmüller spaces*, Comm. Math. Phys. **130** (1990), no. 1, 123–138.

[14] Radnell, D. and Schippers, E. *Quasisymmetric sewing in rigged Teichmller space*, Commun. Contemp. Math. **8** (2006), no. 4, 481534.

[15] Radnell, D. and Schippers, E. *A complex structure on the set of quasiconformally extendible non-overlapping mappings into a Riemann surface*, J. Anal. Math. **108** (2009), 277-291.

[16] Radnell, D. and Schippers, E. *Fiber structure and local coordinates for the Teichmüller space of a bordered Riemann surface*, Conform. Geom. Dyn. **14** (2010), 14–34.

[17] Radnell, D., Schippers, E., and Staubach, W. *Weil-Petersson class non-overlapping mappings into a Riemann surface*, Preprint 2014. (The material of this preprint also appears as part of *A Hilbert manifold structure on the refined Teichmüller space of bordered Riemann surfaces* arXiv:1207.0973.)

[18] Radnell, D., Schippers, E., and Staubach, W. *A Hilbert manifold structure on the Weil-Petersson class Teichmüller space of bordered Riemann surfaces*, Preprint 2014. (The material of this preprint also appears as part of *A Hilbert manifold structure on the refined Teichmüller space of bordered Riemann surfaces* arXiv:1207.0973.)

[19] Radnell, D., Schippers, E., and Staubach, W. *Quasiconformal maps of bordered Riemann surfaces with L^2 Beltrami differentials*, Preprint 2014.

[20] Z. Słodkowski, *Holomorphic motions and polynomial hull*, Proc. Amer. Math. Soc. **111** (1991), no. 2, 347–355.

[21] Strebel, K. *On the maximal dilation of quasiconformal mappings*, Proc. Amer. Math. Soc. **6** (1955), 903–909.

[22] Takhtajan, L. and Teo, L.-P. *Weil-Petersson metric on the universal Teichmller space*, Mem. Amer. Math. Soc. **183** (2006), no. 861.

[23] Vafa, C. *Conformal theories and punctured surfaces*, Phys. Lett. B, **199** no. 2, (1987), 195–202.

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