Conformal welding and the sewing equations

Eric Schippers

Department of Mathematics
University of Manitoba
Winnipeg

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Joint with

David Radnell, American University of Sharjah

and

Wolfgang Staubach, Uppsala Universitet

Our work in general

We began by working on resolving some analytic issues arising in the programme of constructing CFT from VOAs (grew out of David Radnell's PhD thesis under Yi-Zhi Huang)

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David and I discovered that a moduli space in CFT is the quasiconformal Teichmüller space of Ahlfors Bers (Radnell and S, 2005).

A lot of geometric, algebraic and analytic insight can be gained from seeing what the two fields say to each other. This talk is one example.

Conformal welding

Conformal welding in quasiconformal Teichmüller theory

Quasiconformal maps

Definition

A **quasiconformal map** $f : \Sigma \to \Sigma_1$ of Riemann surfaces is a homeomorphism such that

1. f is absolutely continuous on horizontal and vertical lines
2. There is a fixed constant $k < 1$ such that

$$\left| \frac{\partial f}{\partial \bar{z}} \right| \leq k \left| \frac{\partial f}{\partial z} \right|$$

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Quasiconformal maps have a *weaker local condition* but a *stronger global condition* than a C^∞ homeomorphism.

Idea: Quasiconformal maps are distortions of the complex structure.

The Beltrami equation

Theorem (Ahlfors)

Let $\mu d\bar{z}/dz$ be a measurable $(-1,1)$ differential on a Riemann surface Σ. There is a quasiconformal map $f : \Sigma \to \Sigma_1$ such that

$$\overline{\partial} f = \mu \partial f.$$

This map is unique up to composition $f \mapsto \sigma \circ f$ by a biholomoprhism $\sigma : \Sigma_1 \to \Sigma_2$.

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With stronger analytic conditions, this is equivalent to the classical existence of isothermal coordinates.

Quasisymmetries

Definition

A quasisymmetry $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is a map which is the boundary values of a quasiconformal map of $\mathbb{D}$ (or $\mathbb{D}^*$).

$$\mathrm{Diff}(\mathbb{S}^1) \subsetneq \mathrm{QS}(\mathbb{S}^1) \subsetneq \mathrm{Homeo}(\mathbb{S}^1).$$

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Deep question for another talk: what are the correct analytic conditions for parametrizations of boundary curves in CFT?

Conformal welding theorem

Let $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$

Theorem (conformal welding theorem)

Let $\phi \in QS(\mathbb{S}^1)$. There are one-to-one holomorphic maps $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \overline{\mathbb{C}}$ with quasiconformal extensions to $\overline{\mathbb{C}}$ such that

$$\phi = g^{-1} \circ f$$

and $f(0) = 0$, $g(\infty) = \infty$, $g'(\infty) = \alpha$

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Note: it implies conformal welding for analytic or diffeomorphic parametrizations of $\mathbb{S}^1$.
Standard proof: uses existence and uniqueness of solutions to the Beltrami equation.

History of conformal welding

History:

1. First proven by Pfluger (1961).
2. Later (1973) Lehto and Virtanen gave an argument using approximation arguments.
3. Sometimes mistakenly attributed to Kirillov, who gave a proof in 1987 in the diffeomorphism case using the uniformization theorem. Investigating coadjoint orbits of $\mathit{Diff}(\mathbb{S}^1)$.
4. Other proofs in the smooth case exist using approximation arguments (Ebenfeldt and Khavison 2011) or singular integral operators (Gakhov 1966).

Important point: conformal welding plays a central role in quasiconformal Teichmüller theory.

The standard proof

1. Given ϕ, find a quasiconformal map $w_\mu : \mathbb{D}^* \to \mathbb{D}^*$ ($\overline{\partial} w_\mu = \mu \partial w_\mu$) such that $w_\mu|_{\mathbb{S}^1}$.

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3. Let $f = \left. w^\mu \right|_{\mathbb{D}}$ and $g = \left. w^\mu \circ w_\mu^{-1} \right|_{\mathbb{D}^*}$.
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We then have that

$$g^{-1} \circ f\Big|_{\mathbb{S}^1} = w_\mu|_{\mathbb{S}^1} = \phi.$$

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Actually the solution (f, g) is unique (even though there are many possible w_μ).

The sewing equation

Sewing in conformal field theory

Moduli space of Friedan/Shenker/Vafa

Another kind of moduli space appears in CFT.

Let Σ^P be a compact Riemann surface of genus g with n (ordered) punctures.

Definition (Riggings)

A "rigging" is an n-tuple of maps $\phi = (\phi_1, \ldots, \phi_n)$ where $\phi_i : \mathbb{D} \to \Sigma^P$ is a one-to-one, holomorphic map of the unit disc $\mathbb{D}$, taking 0 to the puncture, satisfying

$$\overline{\phi_i(\mathbb{D})} \cap \overline{\phi_j(\mathbb{D})} = \emptyset$$

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Technical but important point: David and I assume that each map ϕ_i has a quasiconformal extension to a neighbourhood of $\overline{\mathbb{D}}$.

Rigged moduli space of Friedan and Shenker

Definition (Rigged moduli space)

The "rigged Riemann moduli space" of type (g, n) is

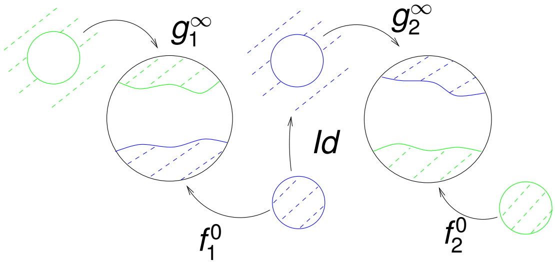
$$\widetilde{\mathcal{M}}(g, n) = \{(\Sigma, \phi)\}/\sim$$

where

- Σ is a compact genus g Riemann surface, with n punctures
- ϕ is a rigging
- $(\Sigma_1, \phi_1) \sim (\Sigma_2, \phi_2)$ if there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $\phi_2 = \sigma \circ \phi_1$.

Explanation of sewing equation

Sew with blue riggings:



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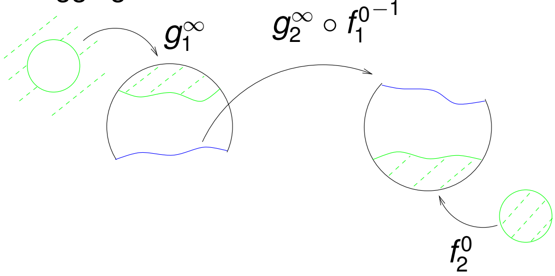

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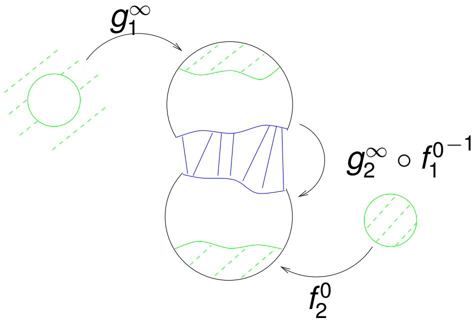
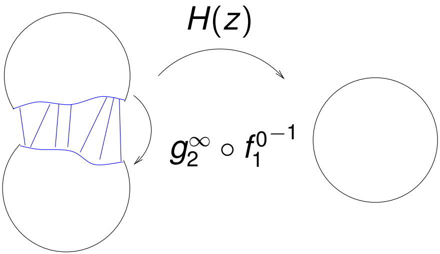


Explanation of sewing equation: uniformize



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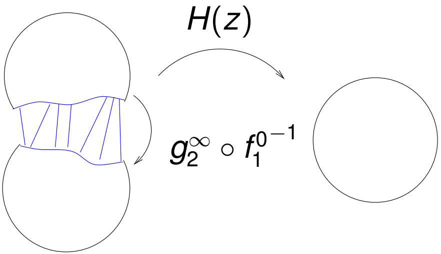

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Sewing equation $F \circ g_2^\infty \circ {f_1}^{0\,-1} = G$**.**

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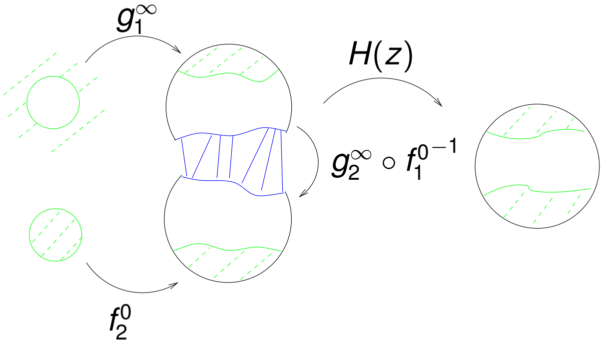

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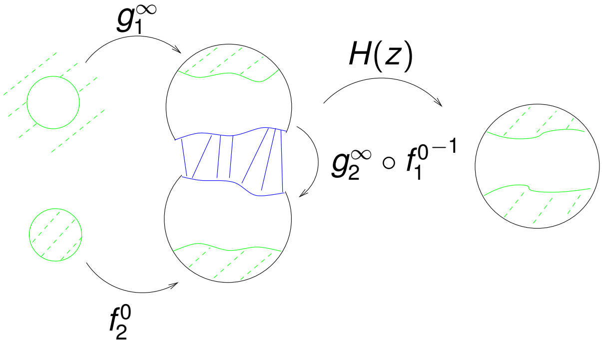

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New riggings: $(F \circ f_2^0, G \circ g_1^\infty)$

Statement of sewing equation

Sewing equation: given one-to-one holomorphic maps f_1 and g_2 on $\mathbb{D}$ and $\mathbb{D}^*$ find F and G such that

$$F \circ g_2 \circ {f_1}^{-1} = G.$$

The proof of the existence of a solution to the sewing equation is due to Y.-Z Huang, [97], for analytic parametrizations.

Quasisymmetric case by Radnell & S (2012) using conformal welding.

Aside on the Teichmüller space/Friedan-Shenker/Vafa moduli space correspondence

The bigger picture for a moment:
Classical fact: the Teichmüller space of $\mathbb{D}$ can be identified with either $\mathrm{QS}(\mathbb{S}^1)/\mathrm{Mb}(\mathbb{S}^1)$ or Friedan-Shenker/Vafa moduli space of spheres with one rigging (though it was not called that!)

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1. The sewing operation on Friedan-Shenker/Vafa moduli space is essential to modularity in CFT.
2. Conformal welding plays a central role in the construction of a complex structure on general quasiconformal Teichmüller spaces.
3. Not a coincidence! since by our work Friedan/Shenker/Vafa $=$ Teichmüller space (up to a discrete group) *for general genus and # of boundary curves*.

New proof of welding

New approach to conformal welding

Algebraic approach to conformal welding?

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First Huang solves the sewing equation formally; convergence is (initially) not an issue. The formal solution involves fairly complicated rings (well, to me anyway...)

Perhaps by adding a little bit of analytic structure first, the algebra simplifies (Teichmüller theory $\to$ CFT). What algebraic insight do we get regarding conformal welding? (CFT $\to$ Teichmüller theory)

The function spaces $\mathcal{H}$ and $\mathcal{H}_*$

Add analytic structure: need a Hilbert space “arena”.
Let $\mathcal{H}$ denote the space of L^2 functions $h = \sum h_n e^{in\theta}$ on $\mathbb{S}^1$ such that

$$\sum_{n=-\infty}^{\infty} |n||h_n|^2 < \infty.$$

Define

$$\|h\|^2 = |\hat{h}(0)|^2 + \sum_{n=-\infty}^{\infty} |n||h_n|^2.$$

We will also consider

$$\mathcal{H}_* = \{h \in \mathcal{H} \ : \ h_0 = 0\}$$

with norm

$$\|h\|_*^2 = \sum_{n=-\infty}^{\infty} |n||h_n|^2.$$

Decomposition of $\mathcal{H}_*$

$$\mathcal{H}_+ = \{h \in \mathcal{H}_* \,:\, h = \sum_{n=1}^{\infty} h_n e^{in\theta}\}$$

$$\mathcal{H}_- = \{h \in \mathcal{H}_* \,:\, h = \sum_{n=-\infty}^{-1} h_n e^{in\theta}\}.$$

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It is well-known that we have the following isometries

$$\begin{aligned} \mathcal{H}_+ &\cong \mathcal{D}(\mathbb{D}) = \{h : \mathbb{D} \to \mathbb{C} \ \text{hol} \ : \iint_{\mathbb{D}} |h'|^2 \, dA < \infty \ \ h(0) = 0\} \\ \mathcal{H}_- &\cong \mathcal{D}(\mathbb{D}^*) = \{h : \mathbb{D}^* \to \mathbb{C} \ \text{hol} \ : \iint_{\mathbb{D}^*} |h'|^2 \, dA < \infty \ \ h(\infty) = 0\} \end{aligned}$$

Summarized in Nag and Sullivan.

Composition operators on $\mathcal{H}$ and $\mathcal{H}_*$

We consider two composition operators

$$C_\phi : \mathcal{H} \to \mathcal{H} \qquad C_\phi h = h \circ \phi$$

$$\hat{C}_\phi : \mathcal{H}_* \to \mathcal{H}_* \qquad C_\phi h = h \circ \phi - \frac{1}{2\pi} \int_{S^1} h \circ \phi(e^{i\theta})\, d\theta.$$

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Theorem (Nag and Sullivan 1993, quoting notes of Zinsmeister)

$\hat{C}_\phi$ is bounded if ϕ is a quasisymmetry.

Theorem (S and Staubach, 2013)

If ϕ is a quasisymmetry then C_ϕ is bounded.

Sketch of a new proof

Treat ϕ as a composition operator C_ϕ on $\mathcal{H}$: we want to solve for unknown functions f and g in equation $f \circ \phi^{-1} = g$, $g_{-1} = \alpha$.

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Using the decomposition $\mathcal{H} = [\mathcal{H}_+] \oplus [\mathbb{C} \oplus \mathcal{H}_-]$, the welding equation can be written

$$C_\phi f = \begin{pmatrix} A & B_{ext} \\ \overline{B}_{ext} & \overline{A}_{ext} \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

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$$Af = g_+ \quad \text{and} \quad \overline{B}_{ext} f = g_-.$$

where $g_+ = g_{-1} z = \alpha z$ and $g_- = g_0 + g_1/z + g_2/z^2 + \cdots$.

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which leads to the solution

$$f = A^{-1} g_+ \quad g_- = \overline{B}_{ext} f.$$

What are the gaps?

We need to show that

- there is a Hilbert space setting and operators are bounded. Done!
- A is invertible: will use symplectic geometry and results of Nag and Sullivan, Takhtajan and Teo.
- The solutions in $\mathcal{H}$ so obtained have the desired properties: conformal with quasiconformal extensions: will use Grunsky inequalities.

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Here we go!

Symplectic structure on $\mathcal{H}_*$

For $x, y \in \mathcal{H}_*$ let

$$\omega(x, y) = -i \sum_{n=-\infty}^{\infty} x_n y_{-n}.$$

If one restricts to the real subspace (such that $x_{-n} = \overline{x_n}$) this is a non-degenerate anti-symmetric form $2\mathrm{Im}\left(\sum_{n=1}^{\infty} \overline{x_n} y_n\right)$.

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Theorem (Nag and Sullivan 1993)

If $\phi : S^1 \to S^1$ is quasisymmetric then $\hat{C}_\phi$ is a symplectomorphism (that is, $\omega(\hat{C}_\phi x, \hat{C}_\phi y) = \omega(x,y)$).

Note that $\hat{C}_\phi$ has the form

$$\begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}.$$

The infinite Siegel disc (Nag and Sullivan)

Definition

The infinite Siegel disc $\mathfrak{S}$ is the set of maps $Z : \mathcal{H}_- \to \mathcal{H}_+$ such that $Z^T = Z$ and $I - Z\overline{Z}$ is positive definite.

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Context:
• the graph of each Z is a Lagrangian subspace of $\mathcal{H}_*$
• symplectomorphisms $\hat{C}_\phi$ act on them.

Definition

Let $\mathcal{L}$ be the set of bounded linear maps of the form

$$(P, Q) : \mathcal{H}_- \to \mathcal{H}_*$$

where $P : \mathcal{H}_- \to \mathcal{H}_+$ and $Q : \mathcal{H}_- \to \mathcal{H}_-$ are bounded operators satisfying $\overline{P}^T P - \overline{Q}^T Q > 0$ and $Q^T P = P^T Q$.

Two facts

- Q invertible $\Rightarrow PQ^{-1} \in \mathfrak{S} \Leftrightarrow (P, Q) \in \mathcal{L}$.
- $(P, Q)Q^{-1} = (PQ^{-1}, I)$ has the same image as (P, Q)

Invariance of $\mathcal{L}$

$\mathcal{L}$ is invariant under bounded symplectomorphisms.

Proposition

If Ψ is a bounded symplectomorphism which preserves $\mathcal{H}_{\mathbb{R}\,*}$ then

$$\Psi \begin{pmatrix} P \\ Q \end{pmatrix} \in \mathcal{L}.$$

Invertibility

Proposition

If $(P, Q) \in \mathcal{L}$ then Q has a left inverse.

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Proof.

If $Q\mathbf{v} = 0$ then

$$0 \leq \overline{\mathbf{v}}^T \left(\overline{Q}^T Q - \overline{P}^T P \right) \mathbf{v} = -\overline{\mathbf{v}}^T \overline{P}^T P\mathbf{v} = -\|P\mathbf{v}\|^2.$$

Thus $Pv = 0$. This implies that $\overline{\mathbf{v}}^T \left(\overline{Q}^T Q - \overline{P}^T P \right) \mathbf{v} = 0$ so by the positive-definiteness of $\overline{Q}^T Q - \overline{P}^T P$, $\mathbf{v} = 0$. Thus Q is injective, or equivalently Q has a left inverse. □

Invertibility of A

Theorem (Takhtajan and Teo 2006; S, Staubach, 2013)

Let $\phi : S^1 \to S^1$ be a quasisymmetry, with

$$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}.$$

Then A is invertible and $Z = B\overline{A}^{-1} \in \mathfrak{S}$.

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Note: This theorem was proven originally by Takhtajan and Teo. However their proof uses the conformal welding theorem, so we must provide a new one.

Proof

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$$\begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} B \\ \overline{A} \end{pmatrix} \in \mathcal{L}.$$

So $\overline{A}$ has a left inverse.

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Apply to ϕ^{-1} (also a quasisymmetry)

$$\hat{C}_\phi = \begin{pmatrix} \overline{A}^T & -B^T \\ -\overline{B}^T & A^T \end{pmatrix}.$$

So $\overline{A}^T$ has a left inverse; thus A is a bounded bijection so it is invertible.

Back to proof of conformal welding theorem

So now we have our f and g.
Problem: They're holomorphic, but how do we know that they're one-to-one (and have quasisymmetric extensions)?

Classical function theory to the rescue: use Grunsky inequalities.

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Classical function theory to the rescue: use Grunsky inequalities.

Let $Z = B\overline{A}^{-1}$.
Recall:

$$(B, \overline{A}) \in \mathcal{L} \Rightarrow B\overline{A}^{-1} \in \mathfrak{S}.$$

Definition of Grunsky matrix

Let

$$g(z) = g_{-1}z + g_0 + g_1 z + g_2 z^2 + \cdots .$$

The Grunsky matrix b_{mn} of g is defined by

$$\log \frac{g(z) - g(w)}{z - w} = \sum_{m,n=1}^{\infty} b_{mn} z^m w^n.$$

Grunsky matrix and welding maps

Theorem (Takhtajan and Teo, 2006)

Let $f(z) = f_1 z + f_2 z^2 + \cdots \in \mathcal{D}(\mathbb{D})$ *and* $g = g_{-1} z + g_{-}$ *where* $g_{-} \in \mathcal{D}(\mathbb{D}^*)$, *and let* $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ *be a quasisymmetry. Assume that* $g \circ \phi = f$ *on* $\mathbb{S}^1$. *Let*

$$\hat{C}_\phi = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \overline{\mathfrak{B}} & \overline{\mathfrak{A}} \end{pmatrix} \quad \text{and} \quad \hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{1}$$

1. *If* $g_{-1} \neq 0$, *then the Grunsky matrix of* g *is* $\overline{B} A^{-1}$.
2. *If* $f_1 \neq 0$, *then the Grunsky matrix of* f *is* $\overline{\mathfrak{B}} \mathfrak{A}^{-1}$.

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2. *If* $f_1 \neq 0$, *then the Grunsky matrix of* f *is* $\overline{\mathfrak{B}} \mathfrak{A}^{-1}$.

Note: Their statement assumes that *f* and *g* are the maps in the conformal welding theorem. However their *proof* only uses the assumptions above and invertibility of *g*.

What Grunsky inequalities do

We have that the Grunsky matrix is

$$\overline{Z} = \overline{B}A^{-1}.$$

where $I - Z\overline{Z}$ is positive definite since $Z \in \mathfrak{S}$. Thus $\|Z\| \leq k < 1$ some k.

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where $I - Z\overline{Z}$ is positive definite since $Z \in \mathfrak{S}$. Thus $\|Z\| \leq k < 1$ some k.

By a classical theorem of Pommerenke if $\|Z\| \leq k < 1$ then g is one-to-one and quasiconformally extendible. A bit of work shows the same for f.

Recap: proof of conformal welding theorem

Proof:
(1) For a quasisymmetry ϕ.

$$\hat{C}_{\phi^{-1}} = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$$

is a symplectomorphism. A is invertible by symplectic linear algebra.

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is a symplectomorphism. A is invertible by symplectic linear algebra.
(2) We may find $f, g \in \mathcal{H}$ such that $f \circ \phi^{-1} = g$ using

$$C_{\phi^{-1}} f = \begin{pmatrix} A & B_{ext} \\ \overline{B}_{ext} & \overline{A}_{ext} \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$$

which has the solution

$$f = A^{-1} g_+ \quad g_- = \overline{B}_{ext} f.$$

Proof continued

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(3) $\overline{Z} = \overline{B}A^{-1}$ is the Grunsky matrix of g under these assumptions, by the theorem of Takhtajan and Teo.

(4) $I - Z\overline{Z}$ is positive definite since $Z \in \mathfrak{S}$. Thus $\|Z\| \leq k < 1$ some k, by symplectic linear algebra.

(5) By classical function theory if $\|Z\| \leq k < 1$ then g and f is one-to-one and quasiconformally extendible.

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for CFT? Given a power series $f(z) = f_1 z + f_2 z^2 + \cdots$, there is a matrix $Z =$ *Grunsky matrix* so that $\|Z\| < 1$ implies f is one-to-one and holomorphic on $\mathbb{D}$.

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Takhtajan and Teo (2006) observed that the Grunsky matrices are in the infinite Siegel disk (=space of polarizations), and that this holomorphically embeds the Teichmüller space of $\mathbb{D}$ in the infinite Siegel disk.
This is a Friedan-Shenker moduli space of rigged disks. By David's and my work, this means the Friedan-Shenker moduli space can be represented this way for higher genus/ n boundary curves.

Work in progress

1. Teichmüller space of genus g with n boundary curves embeds in a Siegel-type space (we're done genus zero case with n boundaries).
2. Determinant line bundle relates to graphs of these operators in a natural way. How do properties of determinant line bundle relate to Grunsky matrices (e.g. sewing, Kähler potential for Weil-Petersson metric)?
3. Construct modular invariants for infinite dimensional Teichmüller spaces.

The end

Thanks!