# THE LOEWNER AND HADAMARD VARIATIONS

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Abstract. We give an explicit formula relating the infinitesimal generators of the Loewner differential equation and the Hadamard variation. This is applied to establish an extension of the Hadamard variation to the case of arbitrary simply–connected domains and to prove the existence of Loewner chains with arbitrary smooth initial generator starting at an arbitrary univalent function which is sufficiently smooth up to the boundary. As another application of this method, we show that every subordination chain $f_t$ is differentiable almost everywhere and satisfies a Loewner equation, without assuming that $f_t'(0)$ is continuous.

## 1. Introduction

The Hadamard (or Julia) variational formula for Green's function is obtained by varying the boundary of a sufficiently smooth domain along its normal by an amount of fixed sign which varies from point to point. One thus obtains a chain of domains of increasing or decreasing size, and a corresponding variational formula. On the other hand, in the case of simply connected domains the Loewner differential equation describes continuously increasing or decreasing families of domains using subordination chains of the corresponding normalized conformal maps. Since Hadamard variation can reach essentially arbitrary nearby domains, it is natural to expect a relation between the two variational methods.

In this paper we relate the Hadamard and Loewner variations in an explicit way in terms of their infinitesimal generators. For instance, we obtain a connection between these two variational methods for the case that the Loewner chain is sufficiently smooth, see Theorem 1 below. For the proof we use a generalization of the Hadamard variational formula to arbitrary homotopies which was recently derived in [9].

This explicit relation between the two variational methods makes it possible to study Hadamard variation from the viewpoint of Loewner's theory and vice versa. In Section 2.2 we focus on one direction of this relation and use the Loewner equation to give a generalization of the Hadamard formula to the case of arbitrary simply-connected domains and a wider class of perturbations. In Section 3 and Section 4, we take the opposite point of view and investigate the Loewner differential equation using Hadamard variation. In Section 3, we derive in this way an existence theorem for the Loewner partial differential equation, which says that given sufficiently smooth initial function $f_0$ and initial generator $p$ of positive real part, there is a Loewner chain $f_t$ so that in the Loewner partial differential equation

$$\frac{\partial f_t}{\partial t}(z) = z p_t(z) \frac{\partial f_t}{\partial z}(z)$$

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we have that $p_0 = p_t$. In other words one can specify not just the initial function (as is well known) but also the initial direction. In the final Section 4 we prove a strengthening of Pommerenke's extension [6, 7] of the Loewner method, where we do not require continuity of the first derivative of the mappings $f_t$. This is achieved by applying a variational formula for Green's function due to Heins [2], which is closely related to Hadamard variation.

## 2. Relation between the Hadamard and Loewner variations

2.1. **Outward variations: the case of smoothly bounded initial domain.** A conformal map $f : \mathbb{D} \to \mathbb{C}$ defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1|\}$ is called a normalized Riemann map if $f(0) = 0$ and $f'(0) > 0$. Let $f_t : \mathbb{D} \to \mathbb{C}$, $t \in [0, T]$, be normalized Riemann maps such that $f_s \prec f_t$ for $0 \le s \le t \le T$ (that is, $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$). Then

$$f_t(z) = \alpha(t)\, z + \dots ,$$

where $\alpha : [0, T] \to (0, \infty)$ is a monotonically increasing function. The subordination chain $f_t$ is called a normalized subordination chain [6] or a Loewner chain [7] if

$$\alpha(t) = e^t .$$

Let $f_t$ be a Loewner chain defined on the interval $[0, T]$ for some $T$.

**Definition 1.** *We say that*

$$F : [a, b] \times [0, L] \to \mathbb{C} \tag{1}$$

*is a "$C^m$ injective homotopy of closed curves" if $F$ is injective on $[a, b]\times[0, L)$, $F(t, 0) = F(t, L)$ for all $t \in [a, b]$, and $F$ has a $C^m$ extension to an open set containing $[a, b] \times \mathbb{R}$ which is $L$ periodic in the second variable. We say that a subordination chain $f_t$ defined on the interval $[a, b]$ is $C^m$ on $[a, b]$ if $f_t$ has a continuous injective extension to $\overline{\mathbb{D}}$ for each $t \in [a, b]$ and the corresponding injective homotopy*

$$F : [a, b] \times [0, 2\pi] \to \mathbb{C}$$
$$(t, \tau) \mapsto f_t(e^{i\tau})$$

*is $C^m$.*

**Remark 1.** *The "closed curves" in the above terminology are of course the curves $\tau \mapsto F(t, \tau)$ for fixed $t$.*

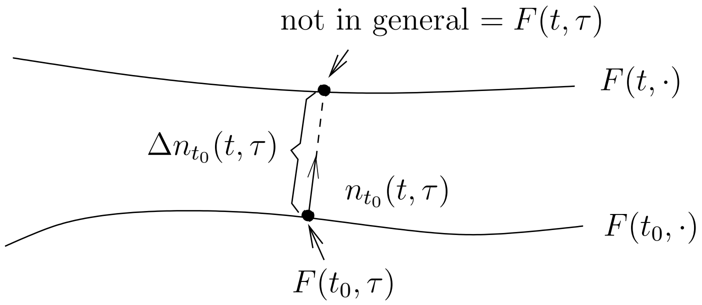
Up to first order, any sufficiently smooth homotopy behaves like a Hadamard variation. To make this precise we need to make some definitions. Consulting Figure 2.1 may be helpful.

**Definition 2.** *Let $F : [a, b] \times [0, L]$ be a $C^2$ injective homotopy of closed curves. Let $n_t(\tau)$ denote the unit outward normal to $F(t, \cdot)$ at $\tau$. For sufficiently small $t - t_0$, let $\Delta n_{t_0}(t, \tau)$ be the distance from $F(t_0, \tau)$ to the curve $F(t, \cdot)$ along the normal $n_{t_0}(\tau)$. Define*

$$\nu_{t_0}(\tau) = \left.\frac{d}{dt}\right|_{t=t_0} \Delta n_{t_0}(t, \tau).$$

It is intuitively clear that for small enough $t - t_0$, $\Delta n_{t_0}(t, \tau)$ is well-defined, and hence $\nu_{t_0}$ is well-defined. Proofs can be found in [9].

FIGURE 1. Definition 2




**Remark 2.** *It will sometimes be convenient to write $\nu_{t_0}(u)$ for $\nu_{t_0}(\tau)$ where $u$ is the complex variable $u = F(t_0,\tau)$ parametrizing the boundary of $f_{t_0}(\mathbb{D})$. Similarly $n(u)$ or $n_u$ will denote the unit outward normal at $u$, etc.*

Let $g_t$ denote Green's function of the domain $f_t(\mathbb{D})$. One would expect that the first-order variation of $g_{t_0}$ should behave as though the homotopy were in fact a variation along the normal lines by the amount $(t-t_0)\nu_{t_0}(\tau)$ at each point $F(t_0,\tau)$. (This is because the variation in the direction tangent to the boundary does not change the domain up to first order). More precisely, we have that [9, Theorem 1]

$$g_t(z,w) - g_{t_0}(z,w) = \frac{1}{2\pi}(t-t_0)\int_{\partial D_t} \frac{\partial g_{t_0}}{\partial n_u}(u,z)\frac{\partial g_{t_0}}{\partial n_u}(u,w)\nu_{t_0}(u)ds_u + O(|t-t_0|^2) \tag{2}$$

where for convenience we denote $\nu_{t_0}(u) = \nu_{t_0}(\tau)$ for $u = F(t_0,\tau)$ along the boundary of $D_{t_0}$, $s_u$ is arc length in the $u$ variable, and $n_u$ denotes the unit outward normal at $u$. The remainder term is understood to be $O(|t-t_0|^2)$ uniformly on compact subsets of $D_{t_0}$ in both $z$ and $w$. Furthermore the remainder term is harmonic. Differentiating

$$\dot{g}_t(z,w) = \frac{1}{2\pi}\int_{\partial D_t} \frac{\partial g_t}{\partial n_u}(u,z)\frac{\partial g_t}{\partial n_u}(u,w)\nu_t(u)ds_u. \tag{3}$$

Let $\mathcal{P}$ denote the set of holomorphic functions $p$ defined on $\mathbb{D}$ satisfying $p(0) = 1$ and $\mathrm{Re}(p) > 0$. We now give an expression for $\nu_t$ in terms of the generator $p_t \in \mathcal{P}$ appearing in the Loewner equation.

**Theorem 1.** *Let $f_t$ be a $C^2$ Loewner chain on $[a,b]$, and let $p_t$ be the infinitesimal generator in the Loewner partial equation*

$$\dot{f}_t(z) = zp_t(z)f_t'(z).$$

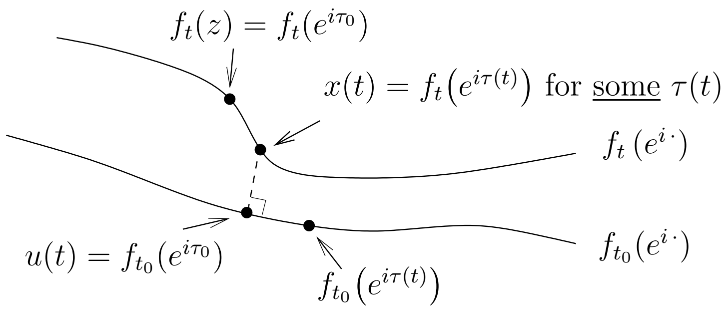
*For $t_0 \in [a,b)$, if $s$ denotes arc length along the boundary of $f_{t_0}(\mathbb{D})$, then for the homotopy $F(t,\tau) = f_t(e^{i\tau})$ we have*

$$\nu_{t_0}(u) = -Re\left(\frac{1}{i}\frac{f_{t_0}^{-1}(u)}{f_{t_0}^{-1\prime}(u)}p_{t_0}\circ f_{t_0}^{-1}(u)\frac{d\bar{u}}{ds}\right).$$

*Proof.* Fix $u \in \partial f_{t_0}(\mathbb{D})$ and let $z = f_{t_0}^{-1}(u)$. Define $x(t) = u + \Delta n_{t_0}(u,t)n_{t_0}(u)$. We claim that

$$\lim_{t\to t_0} \mathrm{Re}\left(\frac{f_t(z)-x(t)}{t-t_0}\overline{n_{t_0}(u)}\right) = 0;$$

FIGURE 2. Proof of Theorem 1




that is,

$$\lim_{t\to t_0} \frac{f_t(z) - x(t)}{t - t_0} \tag{4}$$

is in the direction of the tangent to $\partial D_{t_0}$ at $u(s)$. To see this, by Definition 2

$$\begin{aligned}\lim_{t\to t_0} \frac{x(t) - f_{t_0}(z)}{t - t_0} &= \lim_{t\to t_0} \operatorname{Re}\left(\frac{\Delta n_{t_0}(z,t)}{t - t_0} n_{t_0}(u)\right)\\ &= \nu_{t_0}(u) n_{t_0}(u).\end{aligned}$$

Since

$$\lim_{t\to t_0} \frac{f_t(z) - f_{t_0}(z)}{t - t_0} = \dot{f}_{t_0}(z)$$

it follows that the limit (4) exists. Next, we set $z = e^{i\tau_0}$, and observe that $x(t) = f_t(e^{i\tau(t)})$ for some $\tau(t)$ (see Figure 2.1). Since the homotopy is $C^2$, it follows that

$$\lim_{t\to t_0} \frac{f_t(e^{i\tau_0}) - f_{t_0}(e^{i\tau_0})}{t - t_0} = \lim_{t\to t_0} \frac{f_t(e^{i\tau(t)}) - f_{t_0}(e^{i\tau(t)})}{t - t_0} = \dot{f}_{t_0}(e^{i\tau_0}).$$

Thus by the existence of the limit (4) we may rearrange the terms above to get

$$\lim_{t\to t_0} \frac{f_t(z) - x(t)}{t - t_0} = \lim_{t\to t_0} \frac{f_{t_0}(e^{i\tau_0}) - f_{t_0}(e^{i\tau(t)})}{t - t_0}$$

which is clearly in the direction of the tangent to $\partial f_{t_0}(\mathbb{D})$. This proves the claim.

Thus

$$\begin{aligned}\operatorname{Re}\left(\dot{f}_{t_0}(z)\overline{n_{t_0}(u)}\right) &= \lim_{t\to t_0} \operatorname{Re}\left(\frac{f_t(z) - f_{t_0}(z)}{t - t_0}\overline{n_{t_0}(u)}\right)\\ &= \operatorname{Re}\left(\left(\frac{f_t(z) - x(t)}{t - t_0} + \frac{x(t) - f_{t_0}(z)}{t - t_0}\right)\overline{n_{t_0}(u)}\right)\\ &= \nu_{t_0}(u).\end{aligned}$$

The lemma now follows from the observation that the outward unit normal is given by

$$n_{t_0}(u) = \frac{1}{i}\frac{du}{ds}.$$

□

Thus we have the following extension of the Hadamard variational formula.

**Corollary 1.** *Let $f_t$, $p_t$, etc. be as in Theorem 1. Let $j_t(z) = zp_t(z)$. We have*

$$\dot{g}_t(z,w) = Re\left(\frac{2}{\pi i}\int_{\partial D_t}\frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)j_t\circ f_t^{-1}(u)f_t'\circ f_t^{-1}(u)\,du\right)$$

*Proof.* Since $g$ is constant along the boundary,

$$\mathrm{Im}\left(\frac{1}{i}\frac{\partial g_t}{\partial u}\frac{du}{ds}\right) = 0$$

so

$$\frac{\partial g_t}{\partial n} = \frac{2}{i}\frac{\partial g_t}{\partial u}\frac{du}{ds}.$$

Thus

$$\frac{\partial g_t}{\partial n_u}(u,z)\frac{\partial g_t}{\partial n_u}(u,w) = -4\frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)\left(\frac{du}{ds}\right)^2$$

and this expression is real. The claim follows from (2) and the fact that $|du/ds| = 1$. □

2.2. **Extensions of the Hadamard variational formula.** In this Section we clarify to what extent the Loewner equation for the Riemann map provides an extension of the Hadamard variational formula for Green's function.

We first note that one can easily derive a variational formula for Green's function from the Loewner partial differential equation.

**Theorem 2.** *Let $f_t$ be a Loewner chain on $[0,T]$, and $p_t \in \mathcal{P}$ be the corresponding generator (measurable in $t$) in the Loewner partial differential equation $\dot{f}_t(z) = zp_t(z)f_t'(z)$. If $g_t$ is Green's function of $f_t(\mathbb{D})$ then for almost all $t \in [0,T]$*

(5)
$$\dot{g}_t(z,w) = -2Re\left(\frac{\partial g_t}{\partial z}(z,w)j_t\circ f_t^{-1}(z)f_t'\circ f_t^{-1}(z) + \frac{\partial g_t}{\partial w}(z,w)j_t\circ f_t^{-1}(w)f_t'\circ f_t^{-1}(w)\right).$$

*Proof.* Green's function in terms of $f_t$ is

$$g_t(z,w) = -\log\left|\frac{f_t^{-1}(z) - f_t^{-1}(w)}{1-\overline{f_t^{-1}(w)}f_t^{-1}(z)}\right| \tag{6}$$

Differentiate and apply the Loewner equation. □

Evaluating the integral in Corollary 1 results in the formula above. It is natural to ask whether Theorem 2 can be given in a form closer to the Hadamard variational formula with a suitable interpretation of the integral. This is easily done as follows.

**Theorem 3.** *If $f_t$, $p_t$ and $g_t$ satisfy the hypotheses of Theorem 2 then*

$$\dot{g}_t(z,w) = \lim_{r\to\infty} Re\left(\frac{2}{\pi i}\int_{\gamma_r}\frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)j_t\circ f_t^{-1}(u)f_t'\circ f_t^{-1}(u)\,du\right)$$

*where $\gamma_r$ is the hyperbolic circle of radius $r$ centred on $0$ in $f_t(\mathbb{D})$.*

*Proof.* Since $\partial g/\partial u$ is holomorphic in $u$ with a simple pole at $z$ (resp. $w$), for all $r$ large enough the above integral can be evaluated and equals the expression for $\dot{g}_t$ in Theorem 2. □

It is clear that Theorem 1 also holds for solutions to the inwardly directed Loewner partial differential equation

$$\dot{f}_t = -zp_t(z)f_t'(z), \tag{7}$$

so long as $F(t,\tau) = f_t(e^{i\tau})$ is a $C^2$ injective homotopy. One simply changes the sign of the formula:

$$\nu_{t_0}(u) = \mathrm{Re}\left(\frac{1}{i}\frac{f_{t_0}^{-1}(u)}{f_{t_0}^{-1\prime}(u)}p_{t_0}\circ f_{t_0}^{-1}(u)\frac{d\bar{u}}{ds}\right).$$

Equation (7) was considered by Friedland and Schiffer [3] and is sometimes called the time-reversed Loewner equation or the Friedland-Schiffer equation. They established the existence of solutions for any $p_t \in \mathrm{ext}\mathcal{P}$ measurable in $t$ where

$$\mathrm{ext}\mathcal{P} = \left\{\frac{1+\kappa z}{1-\kappa z}\;\middle|\; |\kappa| = 1\right\}$$

and for any holomorphic initial function $f_0(z)$ on the disc. The solutions are of the form $f_t(z) = f_0(g_t(z))$ where $g_t(z)$ is a bounded univalent function on $\mathbb{D}$ satisfying $g_0(z) = z$ and $g_t'(0) = e^{-t}$. In particular, if $f_0$ is univalent, then $f_t$ can be thought of as an inwardly directed Loewner chain with initial function $f_0$. However, their existence proof does not rely in any way on the fact that $p$ is of the above form, and holds for any $p_t \in \mathcal{P}$ which is measurable in $t$. A proof can also be found in [8].

Theorems 2 and 3 thus clearly hold with a change of sign. That is, let $f_t$ be a solution to equation (7) on the interval $[0,T]$. For almost all $t \in [0,T]$ we have that

(8)

$$\begin{aligned}\dot{g}_t(z,w) &= -\lim_{r\to\infty}\mathrm{Re}\left(\frac{2}{\pi i}\int_{\gamma_r}\frac{\partial g_t}{\partial u}(u,z)\frac{\partial g_t}{\partial u}(u,w)j_t\circ f_t^{-1}(u)f_t'\circ f_t^{-1}(u)\,du\right)\\ &= 2\mathrm{Re}\left(\frac{\partial g_t}{\partial z}(z,w)j_t\circ f_t^{-1}(z)f_t'\circ f_t^{-1}(z) + \frac{\partial g_t}{\partial w}(z,w)j_t\circ f_t^{-1}(w)f_t'\circ f_t^{-1}(w)\right)\end{aligned}$$

where $\gamma_r$ is the hyperbolic circle of radius $r$ centred on 0 in $D_t = f_t(\mathbb{D})$ and $j_t(z) = zp_t(z)$.

**Remark 3.** *For sufficiently smooth solutions to the time-reversed Loewner equation Corollary 1 holds with the opposite sign.*

However, given the existence of solutions to the time-reversed Loewner equation (7) for arbitrary measurable $p_t \in \mathcal{P}$, equation (8) can be stated in a stronger form using the Herglotz representation of $p_t$. Furthermore the quantity $\partial g/\partial n$ can be defined in a natural way for any simply-connected domain by making use of the conformal invariance of Green's function. The idea is to "parametrize" the boundary by hyperbolic angle, and write $\partial g/\partial n$ in terms of the Poisson kernel on $\mathbb{D}$.

Let $f : \mathbb{D} \to D$ be a normalized Riemann map. Then the hyperbolic circle $\gamma_r$ of radius $r$ centred at 0 is given by $\theta \mapsto f(Re^{i\theta})$ for some $R > 0$, and $\theta$ can be interpreted as the hyperbolic angle between the geodesics $f(s)$, $s \in [0,1)$ and $f(se^{i\theta})$, $s \in [0,1)$. Green's function is constant on $\gamma_r$ so for $u = f(Re^{i\theta})$ and any $z \in D$

$$\mathrm{Im}\left(\frac{1}{i}\frac{\partial g}{\partial u}(u,z)\frac{du}{d\theta}\right) = 0$$

so setting $\zeta = Re^{i\theta}$

$$\begin{aligned} \frac{\partial g}{\partial n}(\zeta, z)\frac{ds}{d\theta} &= \frac{2}{i}\frac{\partial g}{\partial u}(\zeta, z)\frac{du}{d\theta} = 2\zeta\frac{\partial g}{\partial u}(\zeta, f^{-1}(z))f'(\zeta) \\ &= 2\zeta\frac{\partial g_{\mathbb{D}}}{\partial \zeta}(\zeta, f^{-1}(z)) \end{aligned} \tag{9}$$

where $g_{\mathbb{D}}$ is Green's function of $\mathbb{D}$. A computation shows that

$$\lim_{R\nearrow 1} 2\zeta\frac{\partial g_{\mathbb{D}}}{\partial \zeta}(Re^{i\theta}, f^{-1}(z)) = -\mathrm{Re}\left(\frac{e^{i\theta}+f^{-1}(z)}{e^{i\theta}-f^{-1}(z)}\right). \tag{10}$$

**Theorem 4.** *Let $D_0$ be a simply connected domain containing 0, and let $\mu_t$ be an increasing function of bounded variation on $[0, 2\pi)$ measurable in $t$ on $[0, T]$, such that $d\mu_t$ has total measure one. Then there exists a family of simply connected domains $D_t$ such that $D_t \subset D_s$ for all $s < t$ whose Green's functions $g_t$ satisfy*

$$\dot{g}_t(z, w) = -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[\frac{e^{i\theta}+f_t^{-1}(z)}{e^{i\theta}-f_t^{-1}(z)}\right] \mathrm{Re}\left[\frac{e^{i\theta}+f_t^{-1}(w)}{e^{i\theta}-f_t^{-1}(w)}\right] d\mu_t(\theta)$$

*for almost all $t$ in $[0, T]$.*

*Proof.* Let $p_t$ be the normalized function of positive real part associated with the measure $d\mu_t$, let $f_t$ be the corresponding solution of the Friedland-Schiffer equation (7) and let $D_t = f_t(\mathbb{D})$. Let $\gamma_r$ be the hyperbolic circle of radius $r$ centred at 0 in $D_t$; so $\gamma_r$ is the image of the Euclidean circle of radius $R$ under $f_t$ for some $R$. By equation (8) for all $r$ large enough we have for $u = f_t(\zeta)$

$$\begin{aligned} \dot{g}_t(z, w) &= -\mathrm{Re}\left(\frac{2}{\pi i}\int_{\gamma_r} \frac{\partial g_t}{\partial u}(u, z)\frac{\partial g_t}{\partial u}(u, w) j_t \circ f_t^{-1}(u) f_t' \circ f_t^{-1}(u)\, du\right) \\ &= -\mathrm{Re}\left(\frac{1}{2\pi i}\int_{f_t^{-1}\circ\gamma_r} \left(\zeta\frac{\partial g}{\partial u}(f_t(\zeta), z)f_t'(\zeta)\right)\left(\zeta\frac{\partial g}{\partial u}(f_t(\zeta), w)f_t'(\zeta)\right) p_t(\zeta)\frac{d\zeta}{\zeta}\right). \end{aligned}$$

By equation (9) the quantities in brackets are real and

$$\begin{aligned} \dot{g}_t(z, w) = -\frac{1}{2\pi}\int_0^{2\pi} &\left(Re^{i\theta}\frac{\partial g}{\partial u}(f_t(Re^{i\theta}), z)f_t'(Re^{i\theta})\right) \\ &\cdot\left(Re^{i\theta}\frac{\partial g}{\partial u}(f_t(Re^{i\theta}), w)f_t'(Re^{i\theta})\right) \mathrm{Re}\, p_t(Re^{i\theta})d\theta. \end{aligned}$$

Now we can choose a sequence $R_n \nearrow 1$ for which the measures $\mathrm{Re}\, p_t(R_n e^{i\,\cdot})$ converge in the weak*–topology to the probability measure $\mu_t$ on $[0, 2\pi]$ associated with $p_t$ via Herglotz formula, and the claim follows from equation (10). □

Theorem 4 is a natural extension of the Hadamard formula (3). In the case that $D_t$ is smoothly bounded and $p_t$ is smooth up to the boundary, it follows from Theorem 1 that if $ds$ denotes infinitesimal arc length then (with $\zeta$, $u$, etc. as above)

$$\nu(u)\frac{d\theta}{ds} = -\mathrm{Re}\, p_t(\zeta).$$

Thus by equation (10)

$$\begin{aligned}\dot{g}_t(z,w) &= -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[\frac{e^{i\theta}+f_t^{-1}(z)}{e^{i\theta}-f_t^{-1}(z)}\right] \mathrm{Re}\left[\frac{e^{i\theta}+f_t^{-1}(w)}{e^{i\theta}-f_t^{-1}(w)}\right] d\mu_t(\theta) \\ &= \frac{1}{2\pi}\int_{\partial D_t}\left(\frac{\partial g}{\partial n_u}(u,z)\frac{ds}{d\theta}\right)\left(\frac{\partial g}{\partial n_u}(u,w)\frac{ds}{d\theta}\right)\nu(u)\frac{d\theta}{ds}d\theta\end{aligned}$$

which agrees with equation (2).

**Remark 4.** *Theorem 3 can also be written in terms of the Herglotz representation of $p_t$. However, given an arbitrary increasing $\mu_t$ of bounded variation and unit total measure there need not be a solution to the Loewner equation (see Example 3 ahead).*

## 3. An Application

3.1. **Existence of solutions to the Loewner partial differential equation with prescribed initial generator.** As an application of Theorem 1, we establish the existence of solutions to the Loewner equation with sufficiently smooth initial infinitesimal generators $p_0 \in \mathcal{P}$.

**Theorem 5.** *Let $f_0 : \mathbb{D} \to D_0$ be a one-to-one and onto holomorphic mapping such that $f_0(0) = 0 \in D_0$. Assume that $f_0 \in C^3(\overline{\mathbb{D}})$, and that the boundary of $D_0$ is a simple curve. For any $p \in \mathcal{P} \cap C^2(\overline{\mathbb{D}})$, there exists a Loewner chain $f_t$ defined on an interval $[0,T]$ satisfying the Loewner partial differential equation*

$$\dot{f}_t = zp_t(z)f_t'(z)$$

*such that $p_0 = p$.*

The proof requires the intuitive geometric fact that for any smooth simple closed curve there exists an interval on which a normal variation is injective.

**Lemma 1.** *Let $\gamma : [0,L] \to \mathbb{C}$ be a $C^2$ simple closed curve with outward normal $n(t)$. Let $K$ be the maximum of the curvature of $\gamma$. Let $d(t_1,t_2) = |\gamma(t_1) - \gamma(t_2)|$, and let $M$ be the minimum of $d$ on*

$$\{\,(t_1,t_2)\,|\,\pi/(5K) \le |t_1 - t_2| \le L/2\,\}.$$

*Finally let $R = \min\{(\sqrt{2}K)^{-1}, M/2\}$. Then, the map $(t,r) \mapsto \gamma(t) + rn(t)$ is injective on $[0,L] \times (-R,R)$.*

*Proof.* (*of Lemma*). By [9, Lemma 2], $(t,r) \mapsto \gamma(t) + rn(t)$ is one-to-one on $[\alpha,\beta] \times (-1/(\sqrt{2}K), 1/(\sqrt{2}K))$ whenever $|\beta - \alpha| < \pi/(4K)$. Now assume that there exist $t_1$ and $t_2$ such that $|t_1 - t_2| \le L/2$ and $\gamma(t_1) + r_1 n(t_1) = \gamma(t_2) + r_2 n(t_2) = w$ for $|r_i| < R$, $i = 1,2$. It follows that $|t_1 - t_2| > \pi/(5K)$. On the other hand, we must also have that

$$\begin{aligned}|\gamma(t_1) - \gamma(t_2)| &\le |\gamma(t_1) - w| + |\gamma(t_2) - w| \\ &= |r_1| + |r_2| < 2R < M\end{aligned}$$

which is a contradiction. □

*Proof.* (*of Theorem*). Let $u(s)$ parametrize $\partial D_0$ by arc length. Let $\nu(u(s))$ be defined by

$$\nu(u(s)) = -\mathrm{Re}\left(f^{-1}(u(s))p\circ f^{-1}(u(s))f'\circ f^{-1}(u(s))\frac{1}{i}\frac{d\bar{u}}{ds}\right).$$

By setting $u(s(t)) = f(e^{it})$ it is easily computed that $\nu(u(s)) > 0$ for all $s$.

Consider the curve $s \mapsto u(s)+\nu(u(s))r$. Since $p$ is in $C^2(\overline{\mathbb{D}})$ and $f' \in C^2(\overline{\mathbb{D}})$, $\nu(u(s))$ is $C^2$ and in particular uniformly bounded on $[0, L]$. Thus since $\partial D_0$ is $C^2$, by Lemma 1 the homotopy $(s, r) \mapsto u(s) + \nu(u(s))r$ is injective on $[0, L] \times [0, T']$ for some $T'$, and furthermore since $\nu(u(s))$ is $C^2$ the homotopy is $C^2$. In particular for each fixed $r$ the resulting curve bounds a simply connected domain $D_r$. Let $\nu_r(s)$ be as in Definition 2. It follows from Theorem 1 that $\nu_0(s) = \nu(u(s))$. Note that this is not true for other values of $r$.

Let $\hat{f}_r : \mathbb{D} \to D_r$ be the conformal mapping such that $\hat{f}_r(0) = 0$ and $\hat{f}_r'(0) > 0$. We claim that the conformal radius $\log|\hat{f}_r'(0)|$ is a $C^1$ function of $r$. To see this, by equation (2) we have

$$\frac{d}{dr}g_r(z, w) = \frac{1}{2\pi}\int_{\partial D_r}\frac{\partial g_r}{\partial n_u}(u, z)\frac{\partial g_r}{\partial n_u}(u, w)\nu_r(u)ds_u$$

where $\nu_r$ is $C^1$ in $r$ and $\partial g_r/\partial n_u$ is $C^2$ in $r$ on $\partial D_r$. Thus

$$\frac{d}{dr}\log|\hat{f}_r'(0)| = \frac{d}{dr}\lim_{z\to 0}(g_r(z, 0) + \log|z|) = \lim_{z\to 0}\frac{d}{dr}(g_r(z, 0) + \log|z|)$$

is $C^1$. Furthermore, since $\nu_0(s) = \nu(u(s))$ one has that the derivative of the conformal radius is 1 at $r = 0$. To see this we proceed as in the proof of Corollary 1:

$$\begin{aligned}\left.\frac{d}{dr}\right|_{r=0}\log|\hat{f}_r'(0)| &= \lim_{z\to 0}\left.\frac{d}{dr}\right|_{r=0}g_r(z, 0)\\ &= \lim_{z\to 0}\mathrm{Re}\left(\frac{2}{\pi i}\int_{\partial D_0}\frac{\partial g_0}{\partial u}(u, 0)\frac{\partial g_0}{\partial u}(u, z)f^{-1}(u)p\circ f^{-1}(u)f'\circ f^{-1}(u)du\right)\\ &= \lim_{z\to 0}\mathrm{Re}\left(p\circ f^{-1}(z)\right) = 1.\end{aligned}$$

Now choose a reparametrization of the subordination chain $f_t = \hat{f}_{r(t)}$ so that $f_t'(0) = e^t$. By the above computation $dr/dt = 1$. Thus

$$\begin{aligned}\left.\frac{d}{dt}g_{r(t)}(z, 0)\right|_{t=0} &= \left.\frac{d}{dr}\right|_{r=0}g_r(z, 0)\\ &= \mathrm{Re}\left(\frac{2}{\pi i}\int_{\partial D_0}\frac{\partial g_0}{\partial u}(u, 0)\frac{\partial g_0}{\partial u}(u, z)f^{-1}(u)p\circ f^{-1}(u)f'\circ f^{-1}(u)du\right)\\ &= \mathrm{Re}\, p\circ f^{-1}(z).\end{aligned}$$

For simplicity we will denote $g_t = g_{r(t)}$; thus $\dot{g}_0(z, 0) = \mathrm{Re}\, p\circ f^{-1}(z)$.

To complete the proof, let $p_t$ be the infinitesimal generator in the Loewner equation for $f_t$. We want to show that $p_0 = p$. Let

$$h_t(z) = -\log\frac{f_t^{-1}(z)}{z}$$

denote the unique choice of analytic completion of $g_t(z,0)+\log|z|$ satisfying $\operatorname{Im} h_t(0) = 0$. By the Loewner equation

$$\dot{h}_0 = -\frac{1}{f_0^{-1}} \left.\frac{d}{dt}\right|_{t=0} f_0^{-1} = p_0 \circ f_0^{-1}$$

is a holomorphic function, whose real part is

$$\operatorname{Re}(\dot{h}_0) = \dot{g}_0 = \operatorname{Re}(p \circ f_0^{-1}).$$

Since $p(0) = p_0(0) = 1$ it follows that $p_0 = p$. □

Theorem 5 also shows that one can arbitrarily prescribe the endpoint and initial generator in the ordinary Loewner equation, so long as these are sufficiently smooth.

**Corollary 2.** *Let $f$ be any univalent function on $\mathbb{D}$ satisfying $f(0) = 0$ and $f'(0) = 1$ such that the boundary of $f(\mathbb{D})$ is $C^3$. Let $p \in \mathcal{P} \cap C^2(\overline{\mathbb{D}})$. There exists a solution to the Loewner ordinary differential equation*

$$\dot{w}_t = -w_t \cdot p_t \circ w_t$$

*with $w_0(0) = 0$, $w_t'(0) = e^{-t}$, and $p_0 = p$, such that*

$$\lim_{t\to\infty} e^t w_t(z) = f(z).$$

*Proof.* By Theorem 5 there exists a Loewner chain $\tilde{f}_t$ defined on $[0,T]$ with initial generator $p_0 = p$. By reparametrizing (and thus possibly changing $T$) one can ensure that $\tilde{f}_t'(0) = e^t$. Let $\hat{f}$ be any normalized Loewner chain on $[T,\infty)$ starting at $\tilde{f}_T$. Defining $f_t = \tilde{f}_t$ for $t \in [0,T]$ and $f_t = \hat{f}_t$ for $t \in (T,\infty)$, we have constructed a normalized Loewner chain on $[0,\infty)$ satisfying the Loewner equation with initial generator $p_0 = p$. Thus $w_t = f_t^{-1} \circ f$ has the desired properties. □

3.2. **Some examples.** It is unclear to what extent the assumptions of Theorem 5 can be weakened. The following examples put some limits on this.

For some choices of initial functions $f_0$, there are $p \in \mathcal{P}$ for which there does not exist a subordination chain on any interval $[0,T]$ so that the initial generator $p_0$ in the Loewner equation is equal to $p$.

**Example 1.** *Let $k(z) = z/(1-z)^2$, $k_t(z) = e^t z/(1-z)^2$ for some $b > 1$ and $f_0(z) = k_t^{-1} \circ k(z)$. For some interval $I = (-1, x_0]$ on the real axis, $f_0$ maps $\mathbb{D}$ onto $\mathbb{D}\backslash I$. Furthermore $f_0$ extends continuously to $\mathbb{D}$, and maps some point $z_0 \in \partial\mathbb{D}$ onto $x_0$. Assume that $p \in \mathcal{P} \cap C^2(\overline{\mathbb{D}})$, and $p \neq 0$ on $f_0^{-1}(J)$ for some open interval $J \subset (-1, x_0)$. It is clear that there is no subordination chain starting at $f_0$ with initial generator $p$.*

It is easy to see that one could find a similar example for which $f_0(\partial\mathbb{D})$ is smooth.

If the boundary of $f_0(\mathbb{D})$ is not smooth, $f^{-1} \cdot p \circ f^{-1} \cdot f' \circ f^{-1}$ need not be continuous even if $p \in \mathcal{P} \cap C^2(\overline{\mathbb{D}})$.

**Example 2.** *Set $w_0 = -(1+i)/2$ and let $f(z) = \sqrt{z+i} + w_0$ where the branch of square root is chosen so that $\mathbb{D}$ is contained in its domain and $\sqrt{i} = (1+i)/\sqrt{2}$. Thus $f$ has a continuous extension mapping $-i$ to a corner of interior angle $\pi/2$ located at $w_0$.*

*It is easily computed that*

$$f^{-1}(w) = (w - w_0)^2 - i \quad \text{and} \quad f' \circ f^{-1}(w) = \frac{1}{2(w - w_0)}.$$

*Setting* $p(z) = 1 + z$ *we have*

$$f^{-1}(w) \cdot p \circ f^{-1}(w) \cdot f' \circ f^{-1}(w) = -\frac{1+i}{2(w - w_0)} + \frac{1-2i}{2}(w - w_0) + \frac{1}{2}(w - w_0)^3.$$

One would expect that for $p \in \mathrm{ext}(\mathcal{P})$ and $f_0(z) = z$, the singularity of $p$ on the boundary would always prevent the existence a subordination chain such that $f_t$ is differentiable in $t$ at $t = 0$ and $p_0 = p$. We have been unable to demonstrate this. However, the following example shows that there is a choice of $p_t$ for $t \in [0, T]$ with $p_t$ in $C^2(\overline{\mathbb{D}})$ on $(0, T]$ and $p_0 \in \mathrm{ext}\mathcal{P}$, for which there is no solution to the Loewner partial differential equation on any interval $[0, T']$ with generator $p_t$ and initial point $f_0(z) = z$.

**Example 3.** *Let*

$$p_t(z) = \frac{1 + e^{-3t}z}{1 - e^{-3t}z}.$$

*Then the (normalized) solution* $f_t$ *to the Loewner equation*

$$\frac{\partial f_t}{\partial t} = z \frac{\partial f_t}{\partial z} p_t$$

*with* $f_0(z) = z$ *is*

$$f_t(z) = 1 - \sqrt{1 - 2e^t z + e^{-2t} z^2}.$$

*This function is not analytic in* $\mathbb{D}$ *for any* $t > 0$.

It should be noted that by result of Becker [1] every solution $f_t$ to the Loewner partial differential equation which is analytic in the disk $|z| < r(t)$ such that $e^t r(t) \to +\infty$ as $t \to +\infty$ is actually analytic in the whole unit disk (see [1, Satz 2]). Thus, if a solution $f_t$ to the Loewner equation does not live on all of $\mathbb{D}$, its domain of definition has to shrink sufficiently fast. This makes it difficult to construct such solutions. Example 3 also shows that the assumption $e^t r(t) \to \infty$ in Becker's result is sharp in a sense.

## 4. A Loewner equation for general subordination chains

In [2] Heins gave an interesting derivation of the Loewner equation. His approach was to first prove that Green's function satisfies a kind of Loewner equation directly, and then use this to derive the Loewner equation for the mapping function. He considered only the special case of Loewner chains of maps onto the disc minus an arc joining the boundary.

In this section, we will show that his approach extends to arbitrary Loewner chains. In fact, this allows the removal of any assumption on the continuity of $f_t'(0)$. We will also show that Heins' formula agrees with Theorem 2 and thus with the Hadamard variational formula.

Recall that a subordination chain is called normalized if $f_t(0) = 0$ and $f_t'(0) = e^t$. It is shown in [6, 7] that a *normalized* subordination chain $f_t$ is differentiable (a.e.) w.r.t. $t$ and that the evolution of $f_t$ can be described with the help of a differential equation (the Loewner equation). The differentiability is based on the fact that every *normalized* subordination chain satisfies a Lipschitz condition w.r.t. $t$ locally uniformly in $\mathbb{D}$.

**Remark 5.** *If $f_t(z) = \alpha(t)z + \cdots$ is a subordination chain such that $\alpha : [0,T] \to (0,\infty)$ is continuous, then the substitution $t^* = \log \alpha(t)$ introduces a new parameter and (w.r.t. to the new parameter) yields a normalized subordination chain $f_{t^*}$, see* [6, 7]. *Thus, if $\alpha(t)$ is continuous, one can select a parametrization which ensures differentiability (a.e.). This is akin to Hilbert's fifth problem, concerning the introduction of differentiable coordinates in continuous groups.*

The following theorem shows that in fact *every* subordination chain is differentiable (a.e.). The proof is based on an idea of Heins [2]. The differentiability and the associated Loewner–type equation come ultimately from the monotonicity of Green's function $g_t$ of $f_t(\mathbb{D})$.

**Theorem 6.** *Let $f_t : \mathbb{D} \to \mathbb{C}$, $t \in [0,T]$, be normalized Riemann maps such that $f_s \prec f_t$ for $0 \le s \le t \le T$. Then there exists a function $p_t(z)$ analytic in $|z| < 1$ and measurable in $t \in [0,T]$ satisfying*

$$\operatorname{Re} p_t(z) \ge 0\,, \qquad z \in \mathbb{D},\, t \in [0,T]$$

*and a set $N \subset [0,T]$ of measure zero such that*

$$\dot{f}_t(z) = z p_t(z)\, f_t'(z)\,, \qquad t \in [0,T] \backslash N\,,\, z \in \mathbb{D}\,.$$

*The map $t \mapsto f_t(\mathbb{D})$ is continuous on $[0,T]\backslash N$ in the sense of kernel convergence.*

*Proof.* (a) Let $A_t := f_t(\mathbb{D})$ and let $g_t(w)$ denote Green's function of $A_t$ with pole at $w = 0$. Note that there exists an open disk $K$ around $w = 0$, which is compactly contained in $A_0$ and thus in every $A_t$. By subordination, $A_s \subset A_t$ for $0 \le s \le t \le T$, so $t \mapsto g_t(w)$ is monotonically increasing for every fixed $w \in K$. If $E := \{w_m\}$, $w_1 := 0$, is a dense countable subset of $K$, then there exists for every nonnegative integer $m$ a nullset $N_m \subseteq [0,1]$ such that $t \mapsto g_t(w_m)$ is differentiable on $[0,T]\backslash N_m$ with derivative $\ge 0$. Thus, for $N := \cup_{m \ge 1} N_m$, the derivative of $g_t(w)$ w.r.t. $t$ exists on $[0,T]\backslash N$ for every $w \in E$.

Then for each $t_0 \in [0,T]\backslash N$ the limit

$$h_{t_0}(w) := \lim_{t \to t_0} \frac{g_t(w) - g_{t_0}(w)}{t - t_0} \tag{11}$$

exists locally uniformly in $K$ and $h_{t_0}$ is a nonnegative and harmonic function in $K$. This follows from the facts that the difference quotients on the right side are nonnegative harmonic functions in $K$ and thus form a normal family, and the limit (11) exists on a dense subset of $K$.

In particular, $g_t(w) + \log|w| \to g_{t_0}(w) + \log|w|$ locally uniformly in $K$ as $t \to t_0$ for every $t_0 \notin N$.

(b) We now show $A_t \to A_{t_0}$ as $t \to t_0$ for any $t_0 \notin N$ in the sense of kernel convergence.

The idea is simply this. We know that $t \mapsto g_t(w)$ is continuous at every $t_0 \notin N$ locally uniformly w.r.t. $w \in K$. Using the relation of Green's function $g_t$ to the conformal map $f_t$, this implies $f_t \to f_{t_0}$ locally uniformly first in a neighborhood of $z = 0$ and then, by normality, in the whole of $\mathbb{D}$, so $A_t \to A_{t_0}$ as $t \to t_0 \notin N$.

Fix $t_0 \not\in N$ and let $G_t$ denote the holomorphic function in $A_t$ with $\operatorname{Re} G_t(w) = g_t(w) + \log|w|$ and $\operatorname{Im} G_t(0) = 0$, that is, by the Schwarz integral formula and shrinking $K$ a little,

$$G_t(w) = \frac{1}{2\pi i} \int\limits_{\partial K} \frac{\zeta + w}{\zeta - w} \big(g_t(\zeta) + \log|\zeta|\big) \frac{d\zeta}{\zeta}, \qquad w \in K. \tag{12}$$

In particular, $G_t \to G_{t_0}$ uniformly in $K$ as $t \to t_0$ and $\dot{G}_{t_0}(w)$ exists uniformly in $w \in K$. Using the relation between Green's function $g_t$ and the conformal map $f_t$,

$$f_t^{-1}(w) = we^{-G_t(w)},$$

we see that $f_t^{-1} \to f_{t_0}^{-1}$ uniformly in $K$ as $t \to t_0$. By the Koebe one–quarter theorem, there is a disk $D \subseteq \mathbb{D}$ such that $D \subset f_t^{-1}(K)$ for all $t \in [0,T]$. It follows that $f_t \to f_{t_0}$ locally uniformly in $D$ as $t \to t_0$. Since $\{f_t : t \in [0,T]\}$ is a normal family, we deduce that $f_t \to f_{t_0}$ locally uniformly in $\mathbb{D}$ as $t \to t_0$, so $f_t(\mathbb{D}) \to f_{t_0}(\mathbb{D})$ as $t \to t_0$ in the sense of kernel convergence.

(c) From (a) and (b) we deduce that for any $t_0 \in [0,T]\backslash N$ the limit (11) exists locally uniformly in $A_{t_0}$ and $h_{t_0}$ is harmonic and nonnegative in $A_{t_0}$. Hence the function $G_{t_0}$ is analytic in $A_{t_0}$, $\dot{G}_{t_0}(w)$ exists locally uniformly for $w \in A_{t_0}$ and

$$\operatorname{Re} \dot{G}_{t_0}(w) = h_{t_0}(w), \qquad w \in A_{t_0}.$$

If $H_{t_0}$ denotes the analytic function in $A_{t_0}$ with $H_{t_0}(0) = \dot{G}_{t_0}(0)$ and $\operatorname{Re} H_{t_0}(w) = h_{t_0}(w) \geq 0$, we therefore get

$$\dot{G}_{t_0}(w) = H_{t_0}(w), \qquad w \in A_{t_0}.$$

Since $f_t^{-1}(w) = we^{-G_t(w)}$, we arrive at

$$\frac{d}{dt}\left(f_t^{-1}\right)\bigg|_{t=t_0}(w) = -H_{t_0}(w)\, f_{t_0}^{-1}(w), \qquad w \in A_{t_0}.$$

Again, the derivative w.r.t. $t$ at $t = t_0$ on the left side exists locally uniformly for $w \in A_{t_0}$. This also implies that $(f_t^{-1})'(w)$ is differentiable w.r.t. $t$ at $t = t_0$ locally uniformly for $w \in A_{t_0}$. By the Bürmann–Lagrange formula,

$$f_t(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\zeta \left(f_t^{-1}\right)'(\zeta)}{f_t^{-1}(\zeta) - z}\, d\zeta, \qquad |z| < r,\ 0 < r < 1, \tag{13}$$

where $\gamma$ is a smooth Jordan curve in $A_t$ which contains $f_t(|\eta| = r)$ in its interior. Thus for fixed $0 < r < 1$, since $f_t(\mathbb{D}) \to f_{t_0}(\mathbb{D})$ as $t \to t_0$, there is a smooth Jordan curve $\gamma \subset A_{t_0}$ which contains $f_t(|\eta| = r)$ in its interior for all $t$ sufficiently close to $t_0$. Hence (13) implies that the limit

$$\dot{f}_{t_0}(z) = \lim_{t \to t_0} \frac{f_t(z) - f_{t_0}(z)}{t - t_0}$$

exists locally uniformly in $\mathbb{D}$. From $f_t^{-1}(f_t(z)) = z$ we therefore get

$$\dot{f}_{t_0}(z) = z f'_{t_0}(z)\, H_{t_0}(f_{t_0}(z)), \qquad z \in \mathbb{D}.$$

If we define

$$p_t(z) := H_t(f_t(z)),$$

then $p_t(z)$ is analytic in $|z| < 1$ with nonnegative real part and measurable in $[0, T]$, and we arrive at the Loewner differential equation for $f_t$. □

**Remark 6.** *This proof does not rely on the second coefficient estimate, distortion theorem or growth theorem for univalent functions. It only relies on the Koebe 1/4 theorem, whose proof also does not require them (see for example* [4]*).*

We conclude with a few observations. First, Theorem 6 generalizes Theorem 2 to the case of an *arbitrary* subordination chain:

**Remark 7.** *Let $D_t$ be any sequence of domains parametrized by $t \in [0, T]$ such that $D_t \subset D_s$ whenever $t < s$. Let $g_t(z, w)$ be Green's function for $D_t$. There exists a function $p_t \in \mathcal{P}$ which is measurable in $t$ and a set $N$ of measure zero such that*

$$\dot{g}_t(z, w) = -2Re\left(\frac{\partial g_t}{\partial z}(z, w) j_t \circ f_t^{-1}(z) f_t' \circ f_t^{-1}(z) + \frac{\partial g_t}{\partial w}(z, w) j_t \circ f_t^{-1}(w) f_t' \circ f_t^{-1}(w)\right),$$

*where $j_t(z) = z p_t(z)$, for any $t \in [0, T] \backslash N$. Furthermore $g_t(z, w) + \log |z - w| \to g_{t_0}(z, w) + \log |z - w|$ locally uniformly for all $t_0 \in [0, T] \backslash N$.*

*Proof.* Differentiate equation (6) using Theorem 6. □

**Remark 8.** *Remark 7 (and thus Theorem 2) agree with Heins' Loewner equation for Green's function. To see this, set $w = 0$ in the above formula. In that case, by equation (6)*

$$\frac{\partial g}{\partial z}(z, 0) = -\frac{{f_t^{-1}}'(z)}{2 f_t^{-1}(z)}.$$

*Thus since $j_t(0) = 0$ and $f_t(0) = 0$ we have*

$$\dot{g}_t(z, 0) = -Re\left(p_t \circ f_t^{-1}(z)\right).$$

*This is Heins' formula (see equations (2) and (3) in* [2]*).*

**Remark 9.** *Theorem 6 shows that the assumption that $d\mu_t$ have unit total measure can be removed from Theorem 4.*

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