

THE SEMIGROUP OF RIGGED ANNULI AND THE TEICHMÜLLER SPACE OF THE ANNULUS

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ABSTRACT. Neretin and Segal independently defined a semigroup of annuli with boundary parametrizations, which is viewed as a complexification of the group of diffeomorphisms of the circle. By extending the parametrizations to quasisymmetries, we show that this semigroup is a quotient of the Teichmüller space of doubly-connected Riemann surfaces by a \mathbb{Z} action. Furthermore, the semigroup can be given a complex structure in two distinct, natural ways. We show that these two complex structures are equivalent, and furthermore that multiplication is holomorphic. Finally, we show that the class of quasiconformally-extendible conformal maps of the disk to itself is a complex submanifold in which composition is holomorphic.

1. INTRODUCTION

1.1. Motivation and statement of results. The Lie algebra of the group of diffeomorphisms of the circle $\text{Diff}(S^1)$ is the Witt algebra. It has been known for some time that there is no Lie group whose Lie algebra is the complexification of the Virasoro algebra or Witt algebra (see Lempert [9] for a proof). Thus there is no Lie group which is the complexification of $\text{Diff}(S^1)$. However, Segal [19] and Neretin [12, 13] independently defined a semigroup which is in some sense the desired complexification. The Neretin-Segal semigroup is defined as follows. The term *annulus* will refer to a bordered Riemann surface that is biholomorphically equivalent to a doubly-connected domain in \mathbb{C} . Consider the set of annuli A , together with parametrizations $\phi_i : S^1 \rightarrow \partial_i A$, $i = 1, 2$, of each boundary component $\partial_i A$. The maps ϕ_1 and ϕ_2 are respectively orientation reversing and orientation preserving. Two such annuli with parametrizations are equivalent if there is a biholomorphism between them which preserves the parametrizations. The multiplication is obtained by sewing the first boundary component of one annulus to the second boundary component of another, by identifying points using the corresponding parametrizations, and carrying along the data of the remaining two parametrizations. From here on we will refer to an element of this semigroup as a *rigged annulus*, where the term “rigging” refers to the boundary parametrizations.

It is customary in conformal field theory (as defined by Segal [19] and Kontsevich) to choose these parametrizations to be either diffeomorphisms or diffeomorphisms with analytic continuations to an annular neighbourhood of the boundary. We choose rather quasisymmetric boundary parametrizations. As a consequence, we are able to prove the following facts.

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Results:

- (1) The quasisymmetric Neretin-Segal semigroup is a quotient of the Teichmüller space of the annulus by a properly discontinuous, fixed-point-free \mathbb{Z} -action. The semigroup of rigged annuli thus inherits a complex structure from the Teichmüller space of the annulus. (Theorem 3.3, Theorem 4.1 and diagram (4.1)).
- (2) The quasisymmetric Neretin-Segal semigroup is a complex Banach manifold, locally modelled on certain function spaces in a natural way. (Theorem 2.20).
- (3) The complex structures in the two previous items are compatible. (Theorem 4.3, Theorem 4.4 and diagram (4.1)).
- (4) Multiplication is holomorphic. (Corollary 4.5).
- (5) The set of quasiconformally extendible one-to-one holomorphic maps of the disk into itself is a complex submanifold of the semigroup of rigged annuli, in which composition is holomorphic. (Theorem 4.8 and Corollary 4.9).

Note that, in items (4) and (5), we prove holomorphicity in the sense that the derivative approximates the function up to first order; this is a much stronger result than Gâteaux holomorphicity. Note also that result (1) establishes that the Teichmüller space of the annulus modulo a \mathbb{Z} action possesses a semigroup structure.

Two points must be emphasized. First, without the choice of quasisymmetric riggings, it is impossible to establish the relation between the Neretin-Segal semigroup and the Teichmüller space. Second, the quasiconformal Teichmüller space of the annulus is infinite-dimensional, and in fact contains the information of the parametrizations. When Teichmüller space appeared in previous models of the rigged moduli spaces of Riemann surfaces of arbitrary type, it was the finite-dimensional Teichmüller space of a compact surface. It appeared as a base space of a fiber space, whose infinite-dimensional fibers consisted of the riggings. The fact that the information of the riggings is somehow contained in the Teichmüller space of a bordered surface was demonstrated in [16].

Using these two insights, in previous work the authors demonstrated the general relation between the rigged moduli spaces and quasiconformal Teichmüller space [16]. Although the result (1) above has never been published, it is an immediate consequence of this previous work. The results (2) and (3) are not, and require some comment. In [18] we used the idea of the fiber space described in the previous paragraph to demonstrate that the Teichmüller space of a bordered Riemann surface has a natural fiber structure, with the fibers consisting of non-overlapping maps into the Riemann surface. This space of non-overlapping maps possesses a complex structure in an independent way (strangely, also related to the universal Teichmüller space) [17]. However, in the case of the annulus, there is a continuous family of conformal automorphisms, and consequently our previous results on the fiber structure do not apply. The same is true for our proofs of the compatibility of the two complex structures. Thus proofs are necessary in the doubly-connected case, and providing them is the main purpose of this paper. On the other hand, in some ways the proof in this special case is more transparent (see Remark 3.13 ahead).

There is growing recognition of the advantages of using quasisymmetries of the circle rather than diffeomorphisms in the literature (e.g. [11, 14, 21]). The quasisymmetric version of the Neretin-Segal semigroup itself appears in Pickrell [15]. We will continue to refer to the semigroup with quasisymmetric riggings as the Neretin-Segal semigroup.

In the rest of Section 1, we define the moduli space of rigged annuli in the quasimetric setting. Section 2 outlines the alternate model of the Neretin-Segal semigroup in terms of non-overlapping mappings into the sphere, which will be the model used throughout the paper. In this section we also prove the multiplication formula in the quasimetric setting, and endow the semigroup with a complex structure. Section 3 proves that the Neretin-Segal semigroup is in one-to-one correspondence with the Teichmüller space of the annulus modulo a \mathbb{Z} action, and thus inherits a complex structure. The main results, the equivalence of the two complex structures, is the subject of Section 4. In this section we also demonstrate the holomorphicity of multiplication and conclude with some consequences for the semigroup of bounded univalent functions with quasiconformal extensions.

1.2. Sewing Riemann surfaces via quasimetrics. In this section we define quasimetric boundary parametrizations and sewing of general Riemann surfaces. Details and proofs can be found in [16].

In the following, the term “bordered” Riemann surface refers to a Riemann surface with boundary in the standard sense (see e.g. [1]). That is, there is an atlas for the Riemann surface such that each point of the boundary is contained in the domain of a chart onto a relatively open subset of the closed upper half plane, which takes the boundary to a finite open subinterval (a, b) of the real line, and furthermore the overlap maps of the atlas are holomorphic on their interiors.

Definition 1.1. We say that Σ is a bordered Riemann surface of type (g, n) if it is a bordered Riemann surface such that (1) its boundary consists of n ordered closed curves homeomorphic to S^1 and (2) it is biholomorphically equivalent to a compact Riemann surface of genus g with n simply-connected non-overlapping regions, each biholomorphic to a disk, removed.

We denote the boundary of Σ by $\partial\Sigma$ and the i th boundary component by $\partial_i\Sigma$. A map $S^1 \rightarrow \partial_i\Sigma$ is called a *boundary parametrization* or *rigging*. Following [16], the class of such mappings will be quasimetric as defined below.

Definition 1.2. An (orientation preserving) homeomorphism

$$h : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$$

is k -quasimetric if there exists a constant k such that

$$\frac{1}{k} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq k$$

for all $x, t \in \overline{\mathbb{R}}$. If h is quasimetric for some unspecified k it is simply called quasimetric.

We find it more convenient to work on S^1 than on $\overline{\mathbb{R}}$. It is also necessary to speak of quasimetricity of a mapping on a closed boundary curve of a Riemann surface. The map $T(z) = i(1+z)/(1-z)$ sends the unit circle to $\overline{\mathbb{R}}$ with $T(1) = \infty$.

For $r \neq 1$, let $\mathbb{A}(r) \subset \mathbb{C}$ be the annulus bounded by circles of radius 1 and r .

Definition 1.3. Let $h : S^1 \rightarrow S^1$ be a homeomorphism.

- (1) Let $e^{i\theta}$ be chosen so that $e^{i\theta}h(1) = 1$. Then we say that h is quasimetric if $T \circ e^{i\theta}h \circ T^{-1}$ is quasimetric according to Definition 1.2.

- (2) Let C be a connected component of the boundary of a bordered Riemann surface, h be a homeomorphism of C into S^1 , and \mathbb{A}_C be an annular neighbourhood of C . We say that h is quasisymmetric if, for any biholomorphism $F : \mathbb{A}_C \rightarrow \mathbb{A}(r)$, $h \circ F^{-1}$ is quasisymmetric on S^1 in the sense of part one.

Note that if $r < 1$ then the above map is orientation preserving, otherwise it is orientation reversing.

A map is quasisymmetric if and only if it is the boundary value of a quasiconformal map on a collared neighbourhood of the boundary. This follows from the Ahlfors-Beurling extension theorem [8].

We now describe the sewing of arbitrary Riemann surfaces. Let Σ_1 and Σ_2 be bordered Riemann surfaces of type (g_1, n_1) and (g_2, n_2) respectively where $n_1 > 0$ and $n_2 > 0$. Let C_1 and C_2 each be a boundary component of Σ_1 and Σ_2 respectively, and let ψ_i be oppositely oriented quasisymmetric parametrizations of C_i (that is, quasisymmetric maps $\psi_i : S^1 \rightarrow C_i$ for $i = 1, 2$, in the sense of Definition 1.3 with say $r_1 > 1$ and $r_2 < 1$). Note that $\psi_2 \circ \psi_1^{-1} : C_1 \rightarrow C_2$ is an orientation reversing map.

Let $\Sigma_1 \# \Sigma_2 = \Sigma_1 \sqcup \Sigma_2 / \sim$ where $x \sim y$ if and only if $x \in C_1$, $y \in C_2$ and $(\psi_2 \circ \psi_1^{-1})(x) = y$. C_1 and C_2 correspond to a common curve on $\Sigma_1 \# \Sigma_2$.

Theorem 1.4. [16, Theorems 3.2 and 3.3] *There is a unique complex structure on $\Sigma_1 \# \Sigma_2$ which is compatible with the original complex structures on Σ_1 and Σ_2 .*

The proof of this theorem is based on conformal welding.

1.3. The moduli space of rigged annuli. In this section we describe the Neretin-Segal semigroup of rigged annuli. In the rest of the paper we will use another model of this moduli space, which will be described in Section 2.1. The model in this section is included because it is more immediately understandable (this is especially true of the multiplication). Both models are well-known in conformal field theory to be equivalent, but we must establish this rigorously in the quasisymmetric setting. This will be done in the next section.

Definition 1.5. Consider the set of ordered pairs (A, ϕ) where

- (1) A is a bordered Riemann surface of type $(0, 2)$ (i.e. doubly-connected) with boundary curves $\partial_i A$, $i = 1, 2$ and
- (2) $\phi = (\phi_1, \phi_2)$ where $\phi_i : S^1 \rightarrow \partial_i A$ are quasisymmetries that are respectively orientation reversing and preserving.

We define

$$\widetilde{\mathcal{M}}(0, 2) = \{(A, \phi)\} / \sim$$

where $(A, \phi) \sim (B, \psi)$ if there exists a biholomorphism $\sigma : A \rightarrow B$ such that $\sigma \circ \phi = \psi$. We denote equivalence classes by $[A, \phi]$.

We have made a slight but fundamental change to the standard definition: the boundary parametrizations are quasisymmetries. As was mentioned in the introduction, our choice makes it possible to connect the moduli space of rigged annuli to the Teichmüller space of the annulus. It is not possible to do this with diffeomorphisms or analytic diffeomorphisms.

Remark 1.6. The rigged moduli space of annuli is a special case of a more general concept from conformal field theory, that of the rigged Riemann surface. Given a bordered Riemann surface Σ of type (g, n) , we denote the ordered set of quasisymmetric boundary

parametrizations by $\psi = (\psi^1, \dots, \psi^n)$. The pair (Σ, ψ) is called a rigged Riemann surface. We define an equivalence relation on the set $\{(\Sigma, \psi)\}$ of type (g, n) rigged Riemann surfaces: $(\Sigma_1, \psi_1) \sim (\Sigma_2, \psi_2)$ if and only if there exists a biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ such that $\psi_2 = \sigma \circ \psi_1$. The moduli space of rigged Riemann surfaces is

$$\widetilde{\mathcal{M}}(g, n) = \{(\Sigma, \psi)\} / \sim .$$

Remark 1.7. In [16] all boundary parametrizations are positively oriented and each boundary component is specified as *incoming* or *outgoing* by an assignment of the symbol $-$ or $+$ respectively. In the current setting this information is specified by the choice of orientation of each rigging, with orientation preserving corresponding to $+$. Hence $\widetilde{\mathcal{M}}(0, 2)$ corresponds to $\widetilde{\mathcal{M}}^B(g, n^-, n^+)$ with $g = n^- = n^+ = 1$ in [16, Definition 5.3].

Next, we describe the multiplication operation in $\widetilde{\mathcal{M}}(0, 2)$.

Definition 1.8. The product of two elements $[A, \phi], [B, \psi]$ of $\widetilde{\mathcal{M}}(0, 2)$ is the rigged annulus

$$[A, \phi] \times [B, \psi] = [A\#B, \rho],$$

where A and B are sewn together along $\partial_1 A$ and $\partial_2 B$ via $\psi_2 \circ \phi_1^{-1}$ and $\rho = (\psi_1, \phi_2)$.

Remark 1.9. One may find a uniformizing biholomorphism $G : A\#B \rightarrow D$ onto an annulus $D \subset \mathbb{C}$; in that case, the joining curve will map to a quasicircle and the new boundary parametrization will be $(G \circ \rho_1, G \circ \rho_2)$.

2. THE NON-OVERLAPPING MAPPING MODEL OF THE SEMIGROUP OF RIGGED ANNULI

2.1. Non-overlapping mappings into the Riemann sphere. In this section, we give an alternate definition of the moduli space of rigged annuli, in terms of non-overlapping mappings into the Riemann sphere. The equivalence of the two models (in the sense that there is a one-to-one correspondence) is well-known [13, 19] in the case of analytic or diffeomorphic parametrizations. We establish here the equivalence with our choice of riggings. The proof of the equivalence relies on the technique of conformal welding.

We must first choose the correct analytic conditions on the set of non-overlapping mappings, to match the quasisymmetric riggings. The obvious choice is the set of univalent maps with quasiconformal extensions. These restrict to quasisymmetries on the boundary, and conversely by the Ahlfors-Beurling extension theorem any quasisymmetry is the boundary value of such a quasiconformally extendible mapping.

We now give the precise description of the alternate model. Let $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$. Note that the boundary $\partial\mathbb{D}^*$ is S^1 with clockwise orientation.

Definition 2.1. Let $\mathcal{A}^o = \{(f, g)\}$ where $f : \mathbb{D} \rightarrow \mathbb{C}$, $g : \mathbb{D}^* \rightarrow \overline{\mathbb{C}}$ are one-to-one holomorphic maps satisfying

- (1) f has a quasiconformal extension to \mathbb{C} and g has a quasiconformal extension to $\overline{\mathbb{C}}$.
- (2) $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \emptyset$
- (3) $f(0) = 0$
- (4) $g(\infty) = \infty$, $g'(\infty) = 1$

It is advantageous to consider an enlargement of this set of annuli, to include “degenerate” annuli whose two boundary components might touch. It is difficult to express this enlargement in terms of Riemann surfaces with boundary parametrizations. However it is easily expressed in terms of non-overlapping holomorphic maps.

Definition 2.2. Let $\mathcal{A} = \{(f, g)\}$ where $f : \mathbb{D} \rightarrow \mathbb{C}$, $g : \mathbb{D}^* \rightarrow \overline{\mathbb{C}}$ are one-to-one holomorphic maps satisfying

- (1) f has a quasiconformal extension to \mathbb{C} and g has a quasiconformal extension to $\overline{\mathbb{C}}$.
- (2) $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \emptyset$
- (3) $f(0) = 0$,
- (4) $g(\infty) = \infty$, $g'(\infty) = 1$

This extension of the rigged annuli has several advantages: first, it has an identity element (it is a monoid), second, it contains the group of quasiconformal maps of S^1 .

In order to show the correspondence between $\widetilde{\mathcal{M}}(0, 2)$ and the Teichmüller space of the annulus, it is necessary to describe the operation of sewing Riemann surfaces using quasiconformal maps. We do this now. Define the punctured closed disks $\overline{\mathbb{D}}_0 = \{z : 0 < |z| \leq 1\}$ and $\overline{\mathbb{D}}_\infty^* = \{z : 1 \leq |z| < \infty\}$, considered as subsets of $\overline{\mathbb{C}}$, and let \mathbb{D}_0 and \mathbb{D}_∞ denote their respective interiors.

Given an annulus A we can obtain a twice punctured genus-zero Riemann surface in the following way. For details and the general case of type (g, n) surfaces, see [16, Section 3]. Let $[A, \tau] \in \widetilde{\mathcal{M}}(0, 2)$ (see Definition 1.5) where $\tau = (\tau_0, \tau_\infty)$, and $\tau_0 : \partial\mathbb{D} \rightarrow \partial_1 A$ and $\tau_\infty : \partial\mathbb{D}^* \rightarrow \partial_2 A$ are fixed quasiconformal mappings (see Definition 1.3). We sew on the punctured disks $\overline{\mathbb{D}}_0$ and $\overline{\mathbb{D}}_\infty^*$ as follows.

Consider the disjoint union of A , $\overline{\mathbb{D}}_0$ and $\overline{\mathbb{D}}_\infty^*$. Identifying boundary points using τ , the result is a compact surface Σ^P with punctures p_0 and p_1 corresponding to the punctures 0 and ∞ of $\overline{\mathbb{D}}_0$ and $\overline{\mathbb{D}}_\infty^*$, respectively. That is, let

$$\Sigma^P = (A \sqcup \overline{\mathbb{D}}_0 \sqcup \overline{\mathbb{D}}_\infty^*) / \sim$$

where $p \sim q$ if and only if $p \in \partial_1 A$, $q \in \partial\mathbb{D}$, and $p = \tau_0(q)$, or $p \in \partial_2 A$, $q \in \partial\mathbb{D}^*$ and $p = \tau_\infty(q)$. By Theorem 1.4, Σ^P has a unique complex structure which is compatible with that of both A and the disks $\overline{\mathbb{D}}_0$ and $\overline{\mathbb{D}}_\infty^*$. If Σ^P is obtained from A in this way we will say that Σ^P is obtained by “sewing caps on A via τ ” and we write

$$(2.1) \quad \Sigma^P = A \#_\tau (\overline{\mathbb{D}}_0 \sqcup \overline{\mathbb{D}}_\infty^*).$$

The parametrizations τ_0 can be extended continuously to a map $\tilde{\tau}_0 : \overline{\mathbb{D}}_0 \rightarrow \Sigma^P$ to the caps of Σ^P by

$$(2.2) \quad \tilde{\tau}_0(x) = \begin{cases} \tau_0(x), & \text{for } x \in \partial\mathbb{D} \\ x, & \text{for } x \in \mathbb{D}_0. \end{cases}$$

This map is a biholomorphism on \mathbb{D}_0 and has a quasiconformal extension to a neighbourhood of $\overline{\mathbb{D}}_0$. Similarly, τ_∞ can be extended to a map $\tilde{\tau}_\infty : \overline{\mathbb{D}}_\infty^* \rightarrow \Sigma^P$. They can also be extended analytically across 0 and ∞ .

We can now make the identification between \mathcal{A}^0 and $\widetilde{\mathcal{M}}(0, 2)$ as follows. Let $[A, \tau] \in \widetilde{\mathcal{M}}(0, 2)$ have representative (A, τ) . Sew on caps following the procedure above to obtain the triple $(\Sigma^P, \tilde{\tau}_1, \tilde{\tau}_2)$. Since Σ^P is a genus zero Riemann surface with two punctures, there

exists a biholomorphism $H : \Sigma^P \rightarrow \mathbb{C}^*$. There is a unique such H such that the conformal extension g of $H \circ \tilde{\tau}_\infty$ satisfies $g'(\infty) = 1$. We then define

$$(2.3) \quad R([A, \tau]) = (f, g)$$

where g is the analytic extension of $H \circ \tilde{\tau}_\infty|_{\mathbb{D}^*}$ across ∞ and f is the analytic extension of $H \circ \tilde{\tau}|_{\mathbb{D}}$ across 0.

Theorem 2.3. $R : \widetilde{\mathcal{M}}(0, 2) \rightarrow \mathcal{A}^0$ is a well-defined, one-to-one, onto map.

Proof. Although a direct proof can be given fairly easily, we refer to [16, Theorem 5.1] with $g = 1$, $n^- = 1$ and $n^+ = 1$ for the sake of brevity (see Remark 1.7). \square

2.2. The sewing equations and multiplication. We now give the formula for multiplication in the non-overlapping mapping model. This formula was obtained by Huang [5]. It is necessary here to give a proof for the class of quasisymmetric riggings, which can be accomplished with the technique of conformal welding. This introduces no difficulties and is no more tedious than the original procedure for sewing, although it does require a deeper uniformization result (ultimately relying on the measurable Riemann mapping theorem).

We will need only the following theorem. It is a standard fact, although usually presented with a different choice of normalizations. Since the normalizations are of some importance, we include a proof.

Theorem 2.4 (conformal welding). *Fix $a \in \mathbb{C} \setminus \{0\}$. Let $\phi : S^1 \rightarrow S^1$ be a quasisymmetric mapping. There exists a unique pair (F, G) , such that $F : \mathbb{D} \rightarrow \mathbb{C}$ and $G : \mathbb{D}^* \rightarrow \overline{\mathbb{C}}$ are one-to-one holomorphic maps with quasiconformal extensions to $\overline{\mathbb{D}}$ satisfying*

- (1) $F(\partial\mathbb{D}) = G(\partial\mathbb{D}^*)$ as sets
- (2) $F(0) = 0$, $G(\infty) = \infty$ and $G'(\infty) = a$
- (3) $\phi = G^{-1} \circ F|_{S^1}$.

Proof. Let $w_\mu : \mathbb{D}^* \rightarrow \mathbb{D}^*$ be a quasiconformal extension of γ , in the sense that it extends homeomorphically to $\overline{\mathbb{D}^*}$ and satisfies $w_\mu|_{S^1} = \phi$. Let μ be the complex dilatation of this extension. Such a quasiconformal extension exists by the Ahlfors-Beurling extension theorem [8]. However, it is not unique.

Let $w^\mu : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be the unique quasiconformal map with dilatation μ on \mathbb{D}^* and 0 on \mathbb{D} , satisfying the normalization $w^\mu(0) = 0$, $w^\mu \circ w_\mu^{-1}(\infty) = \infty$, $(w^\mu \circ w_\mu^{-1})'(\infty) = a$. (w^μ is unique in the sense that it is fixed by μ and these normalizations). Note that $w^\mu \circ w_\mu^{-1}$ is holomorphic. Now set

$$F = w^\mu|_{\mathbb{D}} \quad \text{and} \quad G = w^\mu \circ w_\mu^{-1}|_{\mathbb{D}^*}.$$

Both maps are holomorphic on their domains, have quasiconformal extensions, and clearly $G^{-1} \circ F|_{S^1} = w_\mu|_{S^1} = \phi$. It is also clear that F and G satisfy the desired normalizations.

Now we show that F and G are uniquely determined by ϕ . Given $\phi \in QS(S^1)$ define $\mathcal{T}(\phi) = (F, G)$ where F and G are given by the above procedure. First we show that \mathcal{T} is well-defined. Say w_μ and w_ν are two different quasiconformal extensions of ϕ with dilatations μ and ν respectively. Let

$$S(z) = \begin{cases} w^\mu \circ (w^\nu)^{-1}(z), & \text{if } z \in w^\nu(\mathbb{D}) \\ w^\mu \circ (w_\mu)^{-1} \circ w_\nu \circ (w^\nu)^{-1}(z), & \text{if } z \in w^\nu(\mathbb{D}^*). \end{cases}$$

S is quasiconformal on each piece, and extends to a one-to-one continuous map of $\overline{\mathbb{C}}$ since $w_\mu^{-1} \circ w_\nu$ is the identity on S^1 . Thus S is quasiconformal on $\overline{\mathbb{C}}$, by removability of quasicircles [7, V.3]. Since the dilatation of S is zero on each piece, S is in fact conformal. It is easily checked that $S(0) = 0$, $S(\infty) = \infty$ and $S'(\infty) = 1$, so $S(z) = z$. In particular $w^\mu = w^\nu$ on \mathbb{D} and $w^\mu \circ w_\mu^{-1} = w^\nu \circ w_\nu^{-1}$ on \mathbb{D}^* .

Denote the potential inverse of \mathcal{T} by $\mathcal{S}(F, G) = G^{-1} \circ F$. The second paragraph of the proof shows that \mathcal{S} is surjective. We need to show that \mathcal{S} is injective; to do this we show that $\mathcal{T} \circ \mathcal{S}$ is the identity map. Given (F, G) satisfying the required normalization, let F^μ be a quasiconformal extension of F to the sphere with dilatation μ , say. Then $w_\mu = G^{-1} \circ F^\mu$ is a quasiconformal extension of $G^{-1} \circ F$ to \mathbb{D}^* , and the corresponding w^μ is clearly F^μ . Thus $\mathcal{T} \circ \mathcal{S}(F, G) = (F, G)$. \square

Next we give the formula for multiplication of annuli. The notation here is unfortunately somewhat involved, since we need four conformal maps associated with each element, along with some quasisymmetric riggings.

Let $(f^0, g^\infty) \in \mathcal{A}$ be a non-overlapping pair of conformal maps $f : \mathbb{D} \rightarrow \mathbb{C}$ and $g : \mathbb{D}^* \rightarrow \mathbb{C}$ with quasiconformal extensions. This is a representation of a rigged annulus in the case that $(f^0, g^\infty) \in \mathcal{A}^o$. These two maps f^0 and g^∞ each have ‘complementary’ conformal mapping functions, i.e. the conformal maps onto the complement of their image. We will denote these complementary maps by $f^\infty : \mathbb{D}^* \rightarrow \overline{\mathbb{C}} \setminus \overline{f^0(\mathbb{D})}$ and $g^0 : \mathbb{D} \rightarrow \overline{\mathbb{C}} \setminus \overline{g^\infty(\mathbb{D}^*)}$ respectively. We normalize these maps by requiring $f^\infty(\infty) = \infty$, $(f^\infty)'(\infty) > 0$, $g^0(0) = 0$ and $(g^0)'(0) > 0$.

The element (f^0, g^∞) also has two quasisymmetric mappings corresponding to 0 and ∞ namely $\phi^0 = f^{\infty-1} \circ f^0$ and $\phi^\infty = g^{\infty-1} \circ g^0$. (Conventions: the upper indices of f and g always distinguish whether the mapping is at zero or infinity. For the quasisymmetries, the inverse map on the left is always the one defined at ∞).

Theorem 2.5 (multiplication in \mathcal{A}^0). *Let $(f_1^0, g_1^\infty) \in \mathcal{A}^0$ and $(f_2^0, g_2^\infty) \in \mathcal{A}^0$. Define the two quasisymmetries $\phi_1^0 = f_1^{\infty-1} \circ f_1^0$ and $\phi_2^\infty = g_2^{\infty-1} \circ g_2^0$.*

Let (F, G) be the conformal welding pair such that

$$\phi_1^0 \circ \phi_2^\infty = G^{-1} \circ F,$$

where F and G are normalized by $F(0) = 0, G(\infty) = (\infty), G'(\infty) = (f_1^\infty)'(\infty)$.

The product of the annuli is then given by

$$(2.4) \quad (f_1^0, g_1^\infty) \cdot (f_2^0, g_2^\infty) = \left(F \circ g_2^{0-1} \circ f_2^0, G \circ f_1^{\infty-1} \circ g_1^\infty \right).$$

Proof. Composition preserves quasisymmetries [8], so Theorem 2.4 guarantees the existence of the welding pair (F, G) . According to Definition 1.8 we sew together (f_1^0, g_1^∞) and (f_2^0, g_2^∞) as follows. Remove $f_1^0(\mathbb{D})$ from the first sphere, and $g_2^\infty(\mathbb{D}^*)$ from the second sphere. Join the two remaining domains, identifying points on the boundaries via the map $g_2^\infty \circ f_1^{0-1}$. Denote the new sphere by

$$\mathbb{S} = \overline{\mathbb{C}} \setminus f_1^0(\mathbb{D}) \#_{g_2^\infty \circ f_1^{0-1}} \overline{\mathbb{C}} \setminus g_2^\infty(\mathbb{D}^*).$$

where the complex structure on \mathbb{S} is given by Theorem 1.4. Figure (2.1) below may help visualize the rest of the proof.

The problem now is that this new sphere is an abstract object, and we need to convert it to a standard sphere, and keep track of what happens to the remaining data g_1^∞ and f_2^0 .

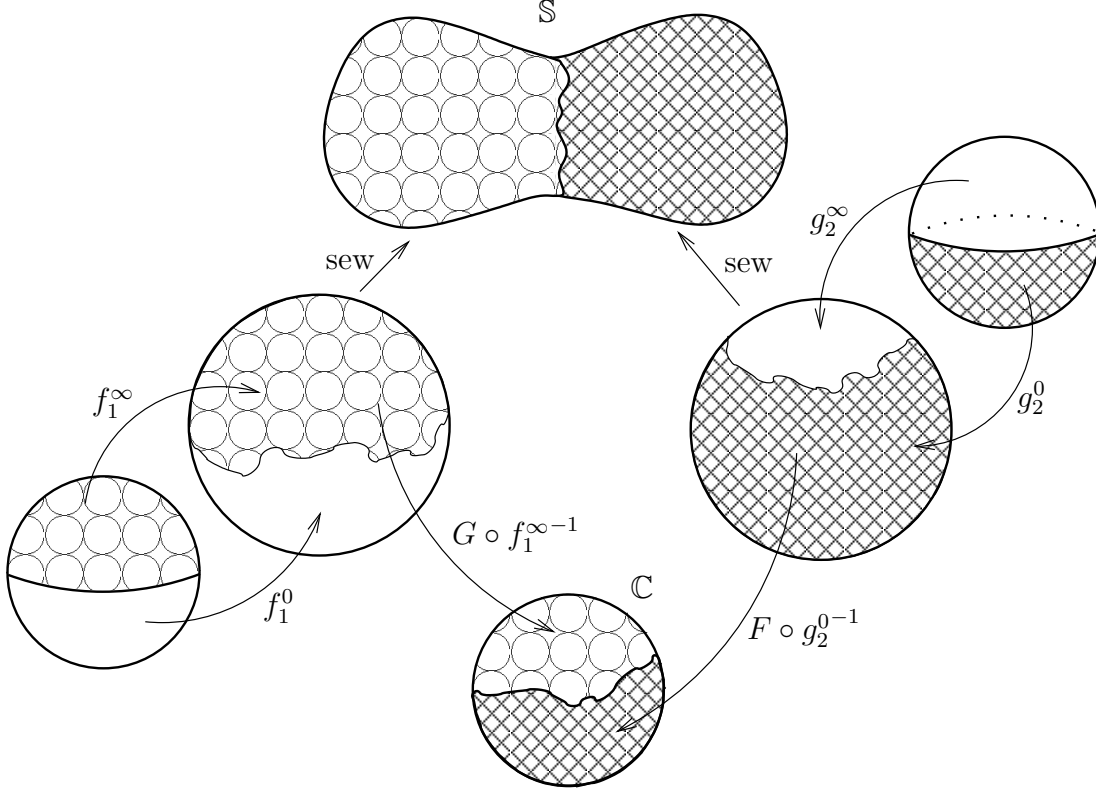


FIGURE 2.1. Sewing spheres using quasisymmetric boundary identification.

To do this, we first give an alternate representation of \mathbb{S} as a join of \mathbb{D}^* to \mathbb{D} (considered as subsets of the spheres corresponding to (f_1^0, g_1^∞) and (f_2^0, g_2^∞) respectively).

These are sewn via $\phi_1^0 \circ \phi_2^\infty = f_1^{\infty-1} \circ f_1^0 \circ g_2^{\infty-1} \circ g_2^0$. That is,

$$\mathbb{S}' \equiv \mathbb{D} \#_{\phi_1^0 \circ \phi_2^\infty} \mathbb{D}^*$$

which is conformally equivalent to \mathbb{S} via the map $\mathbb{S} \rightarrow \mathbb{S}'$ given by

$$z \mapsto \begin{cases} f_1^\infty(z), & \text{if } z \in \mathbb{D}^* \\ g_2^0(z), & \text{if } z \in \mathbb{D}. \end{cases}$$

The final step is to represent \mathbb{S}' as two complementary quasidisks on the standard sphere joined by the identity map.

By Theorem 2.4 there is a unique pair of complementary mappings F and G associated with the quasisymmetry $\phi_1^0 \circ \phi_2^\infty$ such that $\phi_1^0 \circ \phi_2^\infty = G^{-1} \circ F$ and $G'(\infty) = f_1^{\infty-1}(\infty)$. The map which is equal to F on \mathbb{D} and G on \mathbb{D}^* extends continuously to a biholomorphic map from \mathbb{S}' into $\overline{\mathbb{C}}$.

Thus we have that the map which is equal to $F \circ g_2^{0-1}$ on $\overline{\mathbb{C}} \setminus g_2^\infty(\mathbb{D}^*)$ and $G \circ f_1^{\infty-1}$ on $\overline{\mathbb{C}} \setminus f_1^0(\mathbb{D})$ extends continuously to a biholomorphism of \mathbb{S} onto $\overline{\mathbb{C}}$. The new data is obtained by composing this biholomorphism with the riggings g_1^∞ and f_2^0 : at ∞ , we have $G \circ f_1^{\infty-1} \circ g_1^\infty$, and at 0 we have $F \circ g_2^{0-1} \circ f_2^0$. \square

Remark 2.6. The multiplication above extends to \mathcal{A} without difficulty. However, this extended multiplication cannot be considered a consequence of Definition 1.8. The interpretation of the multiplication in the non-overlapping mapping picture does not change.

2.3. Two natural sub-semigroups of \mathcal{A} . \mathcal{A} possesses two natural sub-semigroups [13, 19], in the case that the boundary parametrizations are diffeomorphisms. Allowing the case of degenerate annuli simplifies matters. We include an exposition of the ideas here, both to verify that the picture holds in the case of quasisymmetric and for the convenience of the reader.

Definition 2.7. Let \mathcal{E} denote the subset of \mathcal{A} consisting of elements of the form (f_1^0, Id) . Let $\mathcal{E}^\circ = \mathcal{E} \cap \mathcal{A}^\circ$.

Note that in particular, this implies that $f_1^0 : \overline{\mathbb{D}} \subset \mathbb{D}$ is a bounded univalent function with quasiconformal extension.

Proposition 2.8. \mathcal{E} is a submonoid of \mathcal{A} and \mathcal{E}° is a subsemigroup of \mathcal{A}° . In both cases the multiplication is given by

$$(f_1^0, \text{Id}) \cdot (f_2^0, \text{Id}) = (f_1^0 \circ f_2^0, \text{Id}).$$

Proof. We need to establish that \mathcal{E} is indeed closed under multiplication. If $(f_1^0, \text{Id}), (f_2^0, \text{Id}) \in \mathcal{E}$ then the maps G and F of Theorem 2.5 satisfy $G^{-1} \circ F = f_1^{\infty-1} \circ f_1^0$ since $g_i^\infty = g_i^0 = \text{id}$ for $i = 1, 2$. Thus $G = f_1^\infty$ and $F = f_1^0$, so by the definition of multiplication 2.4 it follows that

$$(2.5) \quad (f_1^0, \text{Id}) \cdot (f_2^0, \text{Id}) = (f_1^0 \circ f_2^0, \text{Id}).$$

It remains to show that \mathcal{E}° is closed under multiplication. This follows from the observations that if $f_1^0(\overline{\mathbb{D}})$ does not intersect S^1 , then neither does $f_1^0 \circ f_2^0(\overline{\mathbb{D}})$, and that a composition of quasiconformal maps is also a quasiconformal map. \square

The set of quasisymmetries of the circle $QS(S^1)$ is a group under composition. One can regard $QS(S^1)$ as consisting of “degenerate annuli” corresponding to welding pairs. That is, if $\phi : S^1 \rightarrow S^1$ is a quasisymmetry we can view the corresponding welding pair (f^0, g^∞) such that $g^{\infty-1} \circ f^0 = \phi$ as an element of \mathcal{A} .

Definition 2.9. Let $\mathcal{G} \subset \mathcal{A}$ denote the set of pairs $(f, g) \in \mathcal{A}$ such that $f(\partial\mathbb{D}) = g(\partial\mathbb{D}^*)$ as sets.

\mathcal{G} is a group and multiplication corresponds to composition of quasisymmetries, as the following proposition shows.

Proposition 2.10. \mathcal{G} is a group, and is isomorphic to $QS(S^1)$ via the map

$$\begin{aligned} \rho : \mathcal{G} &\rightarrow QS(S^1) \\ (f, g) &\mapsto g^{-1} \circ f|_{S^1}. \end{aligned}$$

Proof. Let (f_1^0, g_1^∞) and (f_2^0, g_2^∞) be two such welding pairs corresponding to ϕ_1 and ϕ_2 respectively. Thus we have that $g_i^0 = f_i^0$ and $g_i^\infty = f_i^\infty$ for $i = 1, 2$, and furthermore that $\phi_1^0 = \phi$ and $\phi_2^\infty = \phi_2$ in Definition 2.5. Thus by equation 2.4 it follows that

$$(f_1^0, f_1^\infty) \cdot (f_2^0, f_2^\infty) = (F, G)$$

where $G^{-1} \circ F = \phi_1 \circ \phi_2$, which shows that ρ is a homomorphism. If $\rho(f, g) = g^{-1} \circ f = \text{Id}$, then $f = g = \text{Id}$, so ρ is injective. The fact that ρ is surjective follows directly from Theorem 2.4 with $a = 1$. \square

Remark 2.11. In other words, if $\phi_1 = g_1^{\infty-1} \circ f_1^0$ and $\phi_2 = g_2^{\infty-1} \circ f_2^0$ then

$$\rho((f_1^0, g_1^\infty) \cdot (f_2^0, g_2^\infty)) = \phi_1 \circ \phi_2.$$

2.4. A complex structure on the semigroup of rigged annuli. Let \mathbb{C}^* denote the twice-punctured sphere $\mathbb{C} \setminus \{0\}$. We will now define a natural complex structure on \mathcal{A}^0 . This is inherited from a set of non-overlapping maps, which we now define.

Definition 2.12. Let $\mathcal{O}_{\text{qc}}(\mathbb{C}^*) = \{(f, g)\}$ where (f, g) is a pair of non-overlapping mappings satisfying conditions (1), (2) and (3) in Definition 2.1 and $g(\infty) = \infty$.

The space \mathcal{A}^0 can be identified with a quotient of $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ as follows. Define a \mathbb{C}^* action on $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ by

$$a \cdot (f, g) \mapsto (af, ag)$$

for $a \in \mathbb{C}^*$. This is the action of the automorphism group $\text{Aut}(\mathbb{C}^*) = \{z \mapsto az : a \in \mathbb{C}^*\}$ by composition on the left. The quotient space will be denoted

$$\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$$

with elements denoted $[f, g]$. Each element of $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ has a unique representative in \mathcal{A}^0 . Thus

$$(2.6) \quad \begin{aligned} I : \mathcal{A}^0 &\rightarrow \mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*) \\ (f, g) &\mapsto [f, g] \end{aligned}$$

is a bijection.

Remark 2.13. Although $\text{Aut}(\mathbb{C}^*)$ is the same as \mathbb{C}^* as a set, we will keep distinct notation in the quotient as they have different roles.

We can endow $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ with a complex structure in two distinct ways. We describe one of these ways now. First, we need to define two function spaces.

Definition 2.14. Let \mathcal{O}_{qc} denote the set of $f : \mathbb{D} \rightarrow \mathbb{C}$ satisfying $f(0) = 0$ which are holomorphic, one-to-one and possess a quasiconformal extension to \mathbb{C} .

Definition 2.15. Let

$$A_\infty^1(\mathbb{D}) = \{v(z) : \mathbb{D} \rightarrow \mathbb{C} \mid v \text{ holomorphic, } \|v\|_{1,\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|v(z)| < \infty\}.$$

Note that $A_\infty^1(\mathbb{D})$ is a Banach space. Let $A_\infty^1(\mathbb{D}) \oplus \mathbb{C}$ denote the Banach space with the direct sum norm $\|(\phi, c)\| = \|\phi\|_{1,\infty} + |c|$.

The function space \mathcal{O}_{qc} has a complex structure derived from that of $A_\infty^1(\mathbb{D}) \oplus \mathbb{C}$. Define

$$(2.7) \quad \Psi(f) = \frac{f''}{f'}.$$

The image of \mathcal{O}_{qc} under the map

$$(2.8) \quad \chi(f) = (\Psi(f), f'(0))$$

is an open subset of $A_\infty^1(\mathbb{D}) \oplus \mathbb{C}$ and thus χ induces a complex structure on \mathcal{O}_{qc} [17, Theorem 3.1].

$\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ also inherits a complex structure from $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. It was shown in [17] that the set of non-overlapping quasiconformally extendible conformal maps into a Riemann surface with n distinguished points possesses a complex structure, locally modelled on an n -fold product of \mathcal{O}_{qc} . In particular, $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ can be given a complex structure in this way. In fact, $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ is mapped bijectively onto an open subset of $\mathcal{O}_{\text{qc}} \times \mathcal{O}_{\text{qc}}$ via the map

$$(2.9) \quad \begin{aligned} \theta : \mathcal{O}_{\text{qc}}(\mathbb{C}^*) &\rightarrow \mathcal{O}_{\text{qc}} \times \mathcal{O}_{\text{qc}} \\ (f, g) &\mapsto (f, S(g)) \end{aligned}$$

where

$$(2.10) \quad S(g)(z) = 1/g(1/z).$$

Theorem 2.16. *θ is an injective map onto an open subset of $\mathcal{O}_{\text{qc}} \times \mathcal{O}_{\text{qc}}$. Thus $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ inherits a complex structure from $\mathcal{O}_{\text{qc}} \times \mathcal{O}_{\text{qc}}$.*

Proof. It is obvious that θ is injective. We show that θ is open.

Define $\iota(z) = 1/z$. Fix $(f_0, g_0) \in \mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. Let B^0 and B^∞ be simply connected open sets containing $\overline{f_0(\mathbb{D})}$ and $\overline{g_0(\mathbb{D}^*)}$ such that $B^0 \cap B^\infty$ is empty. Choose $\zeta^0 : B^0 \rightarrow \mathbb{C}$ and $\zeta^\infty : B^\infty \rightarrow \overline{\mathbb{C}} \setminus \{0\}$ to be one-to-one holomorphic maps taking 0 to 0 and ∞ to 0 respectively.

By [17, Corollary 3.5] there is an open neighbourhood U_0 of $\zeta^0 \circ f_0 \in \mathcal{O}_{\text{qc}}$ such that for all $\psi_0 \in U_0$, $\overline{\psi_0(\mathbb{D})} \subset \zeta^0(B^0)$. Similarly, there is a neighbourhood U_∞ of $\zeta^\infty \circ g_0 \circ \iota$ such that for all $\psi_\infty \in U_\infty$, $\overline{\psi_\infty(\mathbb{D})} \subset \zeta^\infty(B^\infty)$. Thus $\theta^{-1}(U_0 \times U_\infty) \subset \mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. This proves that θ is open. \square

Thus we have a global map of $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ into an open subset of a Banach space given by

$$(2.11) \quad \begin{aligned} B : \mathcal{O}_{\text{qc}}(\mathbb{C}^*) &\rightarrow A_\infty^1(\mathbb{D}) \oplus \mathbb{C}^* \oplus A_\infty^1(\mathbb{D}) \oplus \mathbb{C}^* \\ (f, g) &\mapsto (\chi(f), \chi(S(g))) = (\mathcal{A}(f), f'(0), \mathcal{A}(S(g)), g'(\infty)). \end{aligned}$$

We will show how this complex structure passes down to the quotient $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$. Denote for $a \in \mathbb{C}^*$,

$$\mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a = \{(f, g) \in \mathcal{O}_{\text{qc}}(\mathbb{C}^*) : g'(\infty) = a\}.$$

For any a , $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ can be identified with $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a$. Of course $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)_1 = \mathcal{A}^0$, and $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a$ are just cosets of \mathcal{A}^0 under the group action of $\text{Aut}(\mathbb{C}^*) \cong \mathbb{C}^*$ on $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. Observe that

Proposition 2.17. *The action $a \cdot (f, g) = (af, ag)$ of $\text{Aut}(\mathbb{C}^*)$ on $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ is holomorphic.*

Proof. Since $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ is locally modelled on $\mathcal{O}_{\text{qc}} \times \mathcal{O}_{\text{qc}}$, by Hartog's theorem on separate holomorphicity (see [10] for a version in infinite dimensions) it is enough to check that $f \mapsto af$ and $S(g) \mapsto S(g)/a$ are holomorphic maps of \mathcal{O}_{qc} . Since $T(z) = bz$ is a holomorphic map on $\overline{\mathbb{C}}$ this follows immediately from [17, Lemma 3.10]. \square

Define the map

$$(2.12) \quad \begin{aligned} H : \mathcal{O}_{\text{qc}}(\mathbb{C}^*) &\rightarrow \mathbb{C}^* \\ (f, g) &\mapsto g'(\infty). \end{aligned}$$

Theorem 2.18. *H is holomorphic, and possesses a global holomorphic section s .*

Proof. The map $S(g) \mapsto S(g)'(0) = g'(\infty)$ is just χ followed by a projection onto the second component of $A_\infty^1(\mathbb{D}) \oplus \mathbb{C}$, which is clearly holomorphic since χ is. Thus H is holomorphic.

Fix $(f_0, g_0) \in \mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. H has a global section through (f_0, g_0) given by

$$\begin{aligned} s : \mathbb{C}^* &\rightarrow \mathcal{O}_{\text{qc}} \\ a &\mapsto \frac{a}{g_0'(\infty)} \cdot (f_0, g_0). \end{aligned}$$

It is easy to check that $H \circ s(a) = a$ is the identity for any $a \in \mathbb{C}$. Now

$$\mathcal{A}\left(\frac{a}{g_0'(\infty)} f_0\right) = \mathcal{A}(f_0)$$

and similarly for $S(g_0)$. Since

$$a \mapsto B(s(a)) = \left(\mathcal{A}(f_0), a \frac{f_0'(\infty)}{g_0'(\infty)}, \mathcal{A}(S(g_0)), a \right)$$

is holomorphic, it follows that s is holomorphic. \square

Corollary 2.19. *For every $a \in \mathbb{C}^*$, $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a$ is a complex submanifold of $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$.*

Proof. This follows from Proposition 2.17 and Theorem 2.18. See [18, Lemmas 2.15 and 2.16] and references therein. \square

We can transfer the complex structure from $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a$ to $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ as follows. The map

$$\begin{aligned} r_a : \mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*) &\rightarrow \mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a \\ [f, g] &\mapsto \frac{a}{g'(\infty)}(f, g) \end{aligned}$$

is a bijection. By the Corollary, $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ inherits a complex structure from $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ via r_a in which r_a is automatically a biholomorphism.

Combing this fact with Theorem 2.3 and the bijection I in (2.6) we obtain the following important result.

Theorem 2.20. *The two moduli spaces of rigged annuli, $\widetilde{\mathcal{M}}(0, 2)$ and \mathcal{A}^0 inherit complex structures from $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ via the bijections*

$$\widetilde{\mathcal{M}}(0, 2) \xrightarrow{R} \mathcal{A}^0 \xrightarrow{I} \mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*) \xrightarrow{r_a} \mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a \subset \mathcal{O}_{\text{qc}}(\mathbb{C}^*).$$

3. THE RELATION BETWEEN THE MODULI SPACE OF RIGGED ANNULI AND THE TEICHMÜLLER SPACE OF THE ANNULUS

3.1. Teichmüller space and modular groups. Let Σ be a bordered Riemann surface of type (g, n) . Consider the set of triples $\{(\Sigma, f_1, \Sigma_1)\}$ where Σ is a fixed Riemann surface, Σ_1 is another Riemann surface and $f_1 : \Sigma \rightarrow \Sigma_1$ is a quasiconformal map. We say that $(\Sigma, f_1, \Sigma_1) \sim (\Sigma, f_2, \Sigma_2)$ if there exists a biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity “rel boundary”. “Rel boundary” means that the restriction of $f_2^{-1} \circ \sigma \circ f_1$ to the boundary is the identity throughout the homotopy.

Definition 3.1. The Teichmüller space of Σ is

$$T(\Sigma) = \{(\Sigma, f_1, \Sigma_1)\} / \sim .$$

We denote the equivalence classes by $[\Sigma, f_1, \Sigma_1]$.

It is well known that $T(\Sigma)$ is a complex Banach manifold with complex structure compatible with the space $L_{-1,1}^\infty(\Sigma)_1$ of Beltrami differentials via the fundamental projection $\Phi : L_{-1,1}^\infty(\Sigma)_1 \rightarrow T(\Sigma)$. Here $L_{-1,1}^\infty(\Sigma)$ denotes the set of $(-1, 1)$ differentials $\mu d\bar{z}/dz$ with bounded essential supremum, and $L_{-1,1}^\infty(\Sigma)_1$ denotes the unit ball in $L_{-1,1}^\infty(\Sigma)$. The fundamental projection takes Beltrami differentials $\mu d\bar{z}/dz$ to quasiconformal maps whose dilatation is $\mu d\bar{z}/dz$.

As in [16, section 2.1] we introduce a certain subgroup of the mapping class group. Proofs and details can be found there. The pure mapping class group of Σ is the group of homotopy classes of quasiconformal self-mappings of Σ which preserve the ordering of the boundary components. Let $\text{PModI}(\Sigma)$ be the subgroup of the mapping class group consisting of equivalence classes of mappings that are the identity on the boundary $\partial\Sigma$. The group $\text{PModI}(\Sigma)$ is finitely generated by Dehn twists.

The mapping class group acts on $T(\Sigma)$ by $[\rho] \cdot [\Sigma, f, \Sigma_1] = [\Sigma, f \circ \rho, \Sigma_1]$.

Proposition 3.2 ([16, Lemmas 5.1 and 5.2]). *The group $\text{PModI}(\Sigma)$ acts properly discontinuously and fixed-point freely by biholomorphisms on $T(\Sigma)$.*

In the case of an annulus A , $\text{PModI}(A) \simeq \mathbb{Z}$. See [16, Proposition 2.1] for the general case and references. Note that a representative of an element in $\text{PModI}(A)$ is a quasiconformal map $g : A \rightarrow A$ such that $g|_{\partial A} = \text{Id}$. We subsequently identify $\text{PModI}(A)$ with \mathbb{Z} .

The quotient of $T(A)$ by $\text{PModI}(A)$ is in one-to-one correspondence to $\widetilde{\mathcal{M}}(0, 2)$. To exhibit the bijection, fix a rigging τ of the base annulus A . Define

$$(3.1) \quad \begin{aligned} F : T(A)/\text{PModI}(A) &\rightarrow \widetilde{\mathcal{M}}(0, 2) \\ [A, f, A_1] &\mapsto [A_1, f \circ \tau]. \end{aligned}$$

Note that $f \circ \tau$ is a quasisymmetric rigging because the boundary values of the quasiconformal map f are necessarily quasisymmetric.

Theorem 3.3. *The map F is a bijection and hence $\widetilde{\mathcal{M}}(0, 2)$ inherits a complex Banach manifold structure from $T(A)$.*

Proof. This is the special case of [16, Theorem 5.3] with $g = 1, n^- = 1$ and $n^+ = 1$. \square

Remark 3.4. The non-trivial work in Theorem 3.3 is in showing F is onto. It requires the existence of quasiconformal maps between annuli with specified quasisymmetric boundary values, the proof of which ultimately relies on the extended lambda-lemma.

Remark 3.5. The above theorem is in fact not needed in the logical development of this paper because we exhibit three other bijections $K \circ P^{-1}$, R and I which can be combined to give F (see diagram (4.1)). It is included because the explicit map can be easily written.

3.2. Identification of rigged annuli and $T(A)/\mathbb{Z}$. To describe the relation between $T(A)$ and \mathcal{A}^0 we will need the following result.

Lemma 3.6. *Let $[A, h_i, A_i] \in T(A)$ for $i = 1, 2$. Then $[A, h_1, A_1]$ and $[A, h_2, A_2]$ are equivalent in the quotient $T(A)/\mathbb{Z}$ if and only if there exists a biholomorphism $\sigma : A_1 \rightarrow A_2$ such that $h_2 = \sigma \circ h_1$ on ∂A .*

Proof. $h_2 = \sigma \circ h_1$ on ∂A if and only if $h_2^{-1} \circ \sigma \circ h_1 = \text{Id}$ on ∂A if and only if $h_2^{-1} \circ \sigma \circ h_1 \in \text{PModI}(A)$. We previously observed that $\text{PModI}(A) \cong \mathbb{Z}$. \square

To describe the quotient map from $T(A)$ to $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ we will choose the base surface $A \subset \mathbb{C}^*$ and choose a canonical representation of elements of $T(A)$.

Given any doubly-connected bordered Riemann surface A^* and any rigging τ^* , the caps can be sewn on to obtain Σ^P as in equation (2.1) together with the biholomorphically extended rigging $\tilde{\tau}^*$ as in equation (2.2). Since Σ^P has two punctures and genus zero, we may choose a biholomorphism $\sigma : \Sigma^P \rightarrow \mathbb{C}^*$. Let $A = \sigma(\overline{A^*})$, $C_0 = \sigma(\overline{\mathbb{D}_0})$ and $C_\infty = \sigma(\overline{\mathbb{D}_\infty^*})$. The map $\tau = \sigma \circ \tau^*$ is a rigging of A and $\tilde{\tau} = \sigma \circ \tilde{\tau}^*$ is its biholomorphic extension to the caps C_1 and C_2 .

Thus without loss of generality we may choose our base annulus to be an $A \subset \mathbb{C}^*$ which is bounded by quasicircles and give the base annulus a rigging $\tau = (\tau_0, \tau_\infty)$ that extend to biholomorphisms $\tilde{\tau}_0 : \mathbb{D}_0 \rightarrow C_0$ and $\tilde{\tau}_\infty : \mathbb{D}_\infty^* \rightarrow C_\infty$. We henceforth work with such a base surface and rigging.

Definition 3.7. A standard base is (A, τ) where A is a doubly-connected region in \mathbb{C}^* bounded by quasicircles, and $\tau = (\tau_0, \tau_\infty)$ are quasymmetric riggings which extend to biholomorphisms $\tilde{\tau}_0 : \mathbb{D}_0 \rightarrow C_0$ and $\tilde{\tau}_\infty : \mathbb{D}_\infty^* \rightarrow C_\infty$.

Definition 3.8. Let (A, τ) be a standard base. We call a representative (A, h, A') of an element in $T(A)$, a *canonical representative* if A' is a doubly-connected subset of \mathbb{C}^* whose boundaries are quasicircles, and h is the restriction to A of a quasiconformal map $\tilde{h} : \mathbb{C}^* \rightarrow \mathbb{C}^*$ which is a biholomorphism from the complement of \overline{A} to the complement of $\overline{A'}$.

Proposition 3.9. *Let (A, τ) be a standard base. Every element of $T(A)$ has a canonical representative.*

Proof. Choose an arbitrary element $[A, h_1, A_1] \in T(A)$. Sew caps onto A using $\tau = (\tau_0, \tau_\infty)$. Then

$$S = A^* \#_\tau (\overline{\mathbb{D}_0} \sqcup \overline{\mathbb{D}_\infty^*})$$

is biholomorphically equivalent to \mathbb{C}^* via the continuous extension of

$$F(x) = \begin{cases} \tilde{\tau}_0(x), & \text{for } x \in \mathbb{D}_0 \\ x, & \text{for } x \in A \\ \tilde{\tau}_\infty(x), & \text{for } x \in \mathbb{D}_\infty^* \end{cases}$$

where $\tilde{\tau}_0$ and $\tilde{\tau}_\infty$ are the biholomorphic extensions of τ_0 and τ_∞ .

Now sew caps onto A_1 via the rigging $h_1 \circ \tau$ to obtain the Riemann surface

$$S_1 = A_1 \#_{h_1 \circ \tau} (\overline{\mathbb{D}_0} \sqcup \overline{\mathbb{D}_\infty^*})$$

which is biholomorphic to \mathbb{C}^* via some map $\sigma : S_1 \rightarrow \mathbb{C}^*$. Setting $h = F^{-1} \circ \sigma \circ h_1 = \sigma \circ h_1$ and $A' = \sigma(A_1)$ we have $[A, h, A'] \in T(A)$. Here we consider $A_1 \subset S_1$.

Consider the diagram

$$\begin{array}{ccc} S & \xrightarrow{h'_1} & S_1 \\ \downarrow F & & \downarrow \sigma \\ \mathbb{C}^* & \xrightarrow{\tilde{h}} & \mathbb{C}^* \end{array}$$

where h'_1 is defined by

$$(3.2) \quad h'_1(x) = \begin{cases} h_1(x) & \text{for } x \in A \\ x & \text{for } x \in \overline{\mathbb{D}}_0 \sqcup \overline{\mathbb{D}}_\infty^*. \end{cases}$$

and $\tilde{h} = \sigma \circ h'_1 \circ F^{-1}$. That is $\tilde{h} : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is the quasiconformal extension of h given by

$$\tilde{h}(z) = \begin{cases} h = \sigma \circ h_1(z), & \text{for } z \in A \\ \sigma \circ \tilde{\tau}_0^{-1}(z), & \text{for } z \in \tau_0(\mathbb{D}) \\ \sigma \circ \tilde{\tau}_\infty^{-1}(z), & \text{for } z \in \tau_\infty(\mathbb{D}^*). \end{cases}$$

From the definition of the sewing operation, h' is continuous, and hence it is quasiconformal by the removability of quasicarcs for quasiconformal mappings [7, V.3] (see also [16, Section 5.3]). Moreover, the restriction of \tilde{h} to the caps is biholomorphic and so $\tilde{h} \circ \tilde{\tau} \in \mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. Clearly \tilde{h} is conformal on the complement of A and satisfies $\tilde{h}(0) = 0$ and $\tilde{h}(\infty) = \infty$. \square

Definition 3.10. Let (A, τ) be a standard base and let

$$K : T(A) \rightarrow \mathcal{O}_{\text{qc}}(\mathbb{C}^*) / \text{Aut}(\mathbb{C}^*)$$

be defined by

$$K([A, h, A']) = [\tilde{h} \circ \tau_0, \tilde{h} \circ \tau_\infty]$$

where (A, h, A') is a canonical representative and \tilde{h} is the corresponding extension as in Proposition 3.9.

It must be shown that K is well-defined. Assume that (A, h_1, A_1) and (A, h_2, A_2) are canonical representatives which are equivalent in $T(A)$. Then there exists a biholomorphism $\sigma : A_1 \rightarrow A_2$ such that $h_2^{-1} \circ \sigma \circ h_1$ is homotopic to the identity rel boundary. Now σ can be extended to a Möbius transformation as follows. Define

$$\tilde{\sigma}(z) = \begin{cases} \sigma(z) & z \in A_1 \\ \tilde{h}_2 \circ \tilde{h}_1^{-1}(z) & z \in \tau_0(\mathbb{D}) \cup \tau_\infty(\mathbb{D}^*) \end{cases}$$

Since $h_2^{-1} \circ \sigma \circ h_1 = \text{Id}$ on ∂A , this has a continuous extension across the join since $h_2^{-1} \circ \sigma \circ h_1$ is the identity on ∂A , so this map must thus be quasiconformal on $\overline{\mathbb{C}}$. Since it is holomorphic except on a quasicircle, it is in fact holomorphic everywhere [7, V.3]. Thus it is a Möbius transformation which takes 0 to 0 and ∞ to ∞ . So $\tilde{\sigma}(z) = az$, and hence $\sigma \circ h_1 = h_2$ on ∂A . Thus $a\tilde{h}_1 = \tilde{h}_2$ on the complement of A , and so $(a\tilde{h}_1 \circ \tau_0, a\tilde{h}_1 \circ \tau_\infty) = (\tilde{h}_2 \circ \tau_0, \tilde{h}_2 \circ \tau_\infty)$. This shows that K is well-defined.

K is surjective, but fails to be injective. Let $P : T(A) \rightarrow T(A)/\mathbb{Z}$ denote the quotient map. The action by \mathbb{Z} is properly discontinuous and has local holomorphic sections (see Proposition 3.2). It turns out that $K \circ P^{-1}$ is a well-defined map from an open subset of $T(A)/\mathbb{Z}$ to $\mathcal{O}_{\text{qc}}(\mathbb{C}^*) / \text{Aut}(\mathbb{C}^*)$, which we will later show is a biholomorphism onto its image.

Proposition 3.11. *K is surjective. $K([A, h_1, A_1]) = K([A, h_2, A_2])$ if and only if $[A, h_1, A_1]$ and $[A, h_2, A_2]$ are equivalent mod \mathbb{Z} . Thus*

$$K \circ P^{-1} : T(A)/\mathbb{Z} \rightarrow \mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$$

is a well-defined bijection.

Proof. We first show the injectivity of K up to the \mathbb{Z} action. Assume that $K([A, h_1, A_1]) = K([A, h_2, A_2])$. We can assume that (A, h_1, A_1) and (A, h_2, A_2) are canonical representatives. Thus $[\tilde{h}_1 \circ \tau_0, \tilde{h}_1 \circ \tau_\infty] = [\tilde{h}_2 \circ \tau_0, \tilde{h}_2 \circ \tau_\infty]$. So $a\tilde{h}_1 = \tilde{h}_2$ on $\overline{\mathbb{C}} \setminus A$. Since in general (A, ah, aA') is equivalent to (A, h, A') in $T(A)$ we can assume that $A_2 = A_1$ and $\tilde{h}_2 = \tilde{h}_1$ on $\overline{\mathbb{C}} \setminus A$. In particular, h_1 and h_2 agree on ∂A , so by Lemma 3.6 they are equivalent mod \mathbb{Z} .

Conversely, assume that $[A, h_1, A_1]$ and $[A, h_2, A_2]$ are equivalent mod \mathbb{Z} . Then the extensions \tilde{h}_1 and \tilde{h}_2 agree on $\overline{\mathbb{C}} \setminus A$ by Lemma 3.6 and so $K([A, h_1, A_1]) = K([A, h_2, A_2])$.

Finally, we demonstrate that K is onto. By [16, Corollary 4.1] for any $(f, g) \in \mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ there is a quasiconformal extension $\tilde{h} : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ of $f \circ \tau_0^{-1}$ and $g \circ \tau_\infty^{-1}$. Set $A' = \tilde{h}(A)$ and we have that $K([A, h, A']) = (f, g)$. \square

In proving the surjectivity of K , we very nearly defined the inverse of $K \circ P^{-1}$.

Proposition 3.12. *The inverse of $K \circ P^{-1}$ is*

$$L : \mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*) \rightarrow T(A)/\mathbb{Z}$$

$$[f, g] \mapsto (A, \tilde{h} \Big|_A, \tilde{h}(A))$$

where \tilde{h} is a quasiconformal extension to \mathbb{C}^ of $f \circ \tau_0^{-1}$ and $g \circ \tau_\infty^{-1}$ for some representative (f, g) of $[f, g]$.*

Proof. The last paragraph of the proof of Proposition 3.11 shows that L is a right inverse of $K \circ P^{-1}$, so long as it is well-defined.

To show that L is well-defined, assume that \tilde{h}_1 and \tilde{h}_2 are two possibly distinct quasiconformal extensions of $f \circ \tau_0^{-1}$ and $g \circ \tau_\infty^{-1}$, for a fixed representative (f, g) of $[f, g]$. In this case \tilde{h}_1 and \tilde{h}_2 must agree on ∂A , so $A_2 = A_1$ and (A, h_1, A_1) and (A, h_2, A_1) are in the same equivalence class mod \mathbb{Z} by Lemma 3.6. So L is independent of the choice of extension.

Now assume that $[f_1, g_1] = [f_2, g_2]$. Then $f_2 = af_1$ and $g_2 = ag_1$ for some $a \in \mathbb{C}^*$. If \tilde{h}_i are the corresponding extensions of $f_i \circ \tau_0^{-1}$ to $\overline{\mathbb{C}}$ for $i = 1, 2$ and setting $A_i = \tilde{h}_i(A)$, then $aA_1 = A_2$ and $a\tilde{h}_1 = \tilde{h}_2$ on ∂A . By Lemma 3.6

$$[A, \tilde{h}_1 \Big|_A, A_1] = [A, \tilde{h}_2 \Big|_A, A_2].$$

Thus L is well-defined.

Since $K \circ P^{-1}$ is injective, and L is a right inverse, it is also a left inverse. So $K \circ P^{-1} = L^{-1}$. \square

Remark 3.13. The maps L and $K \circ P^{-1}$ are analogous to the map identifying the set of non-overlapping maps $\mathcal{O}_{\text{qc}}(\Sigma)$ with a fiber in $T(\Sigma)$ which we defined in [18]. However the results of that paper do not apply in this case, because there we used the fact that the automorphism group has no continuous subgroups in an essential way; this is false for an annulus. Furthermore, the base space reduces to a point, and the ‘‘fiber’’ becomes rather

$\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$. On the other hand, in some ways the proof that L is a biholomorphism is more transparent in the case at hand, because here L has an *explicit* inverse.

4. THE COMPLEX STRUCTURES AND HOLOMORPHICITY OF MULTIPLICATION

4.1. The complex structure inherited from $T(A)$. Since $\text{PModI}(A) \cong \mathbb{Z}$ acts properly discontinuously and fixed-point freely by biholomorphisms, it follows from Proposition 3.11 that \mathcal{A}^0 inherits a complex structure from that of $T(A)$. We have thus shown that

Theorem 4.1. \mathcal{A}^0 possesses a complex structure inherited from $T(A)$.

Furthermore, by our previous work, in this complex structure, the multiplication is holomorphic.

Theorem 4.2. Multiplication in \mathcal{A}^0 in the complex structure inherited from $T(A)$ is holomorphic.

Proof. This follows from [16, Theorem 6.7], with the choice $g_X = g_Y = 0$, $n_X^- = n_Y^- = 1$, $n_X^+ = n_Y^+ = 1$. \square

4.2. Compatibility of the two complex structures. To show that the two complex structures on $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ are compatible, we need to show that the maps L and $L^{-1} = K \circ P^{-1}$ are holomorphic, where $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*)$ is endowed with the complex structure inherited from $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$.

Theorem 4.3. L is holomorphic.

Proof. L has a lift to $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ given by

$$\begin{aligned} \tilde{L} : \mathcal{O}_{\text{qc}}(\mathbb{C}^*) &\rightarrow T(A)/\mathbb{Z} \\ (f, g) &\mapsto (A, \tilde{h} \Big|_A h(A)) \end{aligned}$$

where \tilde{h} is a quasiconformal extension of $f \circ \tau_0^{-1}$ and $g \circ \tau_\infty^{-1}$ (one exists by [16, Corollary 4.1]). It follows directly that $L = \tilde{L} \circ r_a$. Since r_a is biholomorphic, it suffices to show that \tilde{L} is holomorphic in order to show that L is. It furthermore suffices to show that in a neighbourhood of any point $P^{-1} \circ \tilde{L}$ is holomorphic for some local inverse of P . The fundamental projection $\Phi : L_{-1,1}^\infty(A) \rightarrow T(A)$ has local holomorphic inverses in a neighbourhood of any point. We will show that for any choice of local inverses Φ and P , $\Phi^{-1} \circ P^{-1} \circ \tilde{L}$ is Gâteaux holomorphic and locally bounded. This is sufficient to demonstrate holomorphicity [3, p 178].

Fix a point $(f, g) \in \mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. Let $B(f, g) = (u^0, q^0, u^\infty, q^\infty)$ where B is the map defined by equation (2.11). We will construct a holomorphic curve through $B(f, g)$. Let $(v^0, c^0, v^\infty, c^\infty) \in A_\infty^1(\mathbb{D}) \oplus \mathbb{C}^* \oplus A_\infty^1(\mathbb{D}) \oplus \mathbb{C}^*$. Define the complex line $Y(t) = (u_t^0, q_t^0, u_t^\infty, q_t^\infty)$ by

$$Y(t) = (u^0, q^0, u^\infty, q^\infty) + t(v^0, c^0, v^\infty, c^\infty).$$

The curve $B^{-1} \circ Y(t)$ has an explicit expression, which can be found by integrating the differential equation $\mathcal{A}(f_t) = u^0 + tv^0$, $f_t'(0) = q_t^0$, and similarly for g_t . The solution is

$$\psi_t = (f_t, g_t)$$

where

$$f_t = \frac{q_t^0}{q^0} \int_0^z f'(\xi) \exp\left(t \int_0^\xi v^0(w) dw\right) d\xi$$

and

$$S(g_t) = \frac{q_t^\infty}{q^\infty} \int_0^z S(g)'(\xi) \exp\left(t \int_0^\xi v^\infty(w) dw\right) d\xi.$$

We can recover g_t from the second expression if desired by using the fact that $S(S(g_t)) = g_t$.

For fixed z , both expressions are holomorphic in t by inspection. By [17, Theorem 3.3] there is a neighbourhood N of the origin in \mathbb{C} such that (f_t, g_t) are in $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ for all $t \in N$. Thus the map of $f_0(\mathbb{D}) \cup g_0(\mathbb{D}^*)$ given by $f_t \circ f_0^{-1}$ restricted to $f_0(\mathbb{D})$ and $g_t \circ g_0^{-1}$ restricted to $g_0(\mathbb{D}^*)$ is a holomorphic motion. By the extended lambda-lemma [20], this extends to a holomorphic motion \tilde{h}_t of $\overline{\mathbb{C}}$. Thus denoting $h_t = \tilde{h}_t|_A$ and $A_t = h_t(A)$, we get a curve

$$t \mapsto (A, h_t, A_t)$$

in $T(A)$.

Because h_t is a holomorphic motion, its complex dilatation $\mu(h_t)$ is a holomorphic curve in $L_{-1,1}^\infty(A)$ (see for example [2, Theorem 2]). Clearly $\mu(h_t) = \Phi^{-1} \circ P^{-1} \circ \tilde{L}(f_t, g_t)$ for some local choices of Φ^{-1} and P^{-1} . We have thus shown that $\Phi^{-1} \circ P^{-1} \circ \tilde{L}$ is Gâteaux holomorphic for some local choices of P^{-1} and Φ^{-1} .

Since $L_{-1,1}^\infty(A)_1$ is globally bounded by one, $\Phi^{-1} \circ P^{-1} \circ \tilde{L}$ is locally bounded. This proves the claim. \square

Theorem 4.4. L^{-1} is holomorphic.

Proof. It is enough to show that K is holomorphic, since $P : T(A) \rightarrow T(A)/\mathbb{Z}$ has local holomorphic sections and $L^{-1} = K \circ P^{-1}$. K has a lift

$$\begin{aligned} \tilde{K} : T(A) &\rightarrow \mathcal{O}_{\text{qc}}(\mathbb{C}^*) \\ [A, h, A'] &\mapsto (\tilde{h} \circ \tau_0, \tilde{h} \circ \tau_\infty) \end{aligned}$$

where we choose the unique representation $(\tilde{h} \circ \tau_0, \tilde{h} \circ \tau_\infty)$ of $K([A, h, A'])$ with the normalization $(\tilde{h} \circ \tau_\infty)'(\infty) = 1$. It suffices to show that \tilde{K} is holomorphic.

Fix $[A, h, A'] \in T(A)$, and let $(f, g) = \tilde{K}([A, h, A'])$. Let B^0 and B^∞ be open sets containing $\overline{f(\mathbb{D})}$ and $\overline{g(\mathbb{D}^*)}$ respectively. Let V be a neighbourhood of $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$ such that $\overline{f_1(\mathbb{D})} \subset B^0$ and $\overline{g_1(\mathbb{D}^*)} \subset B^\infty$ for all $(f_1, g_1) \in V$. Recall that the map $(f_1, g_1) \mapsto (f_1, S(g_1))$ is holomorphic by definition of the complex structure on $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. Let $\pi_1 : \mathcal{O}_{\text{qc}}(\mathbb{C}^*) \rightarrow \mathcal{O}_{\text{qc}}$ be the projection onto the first component $(f_1, g_1) \mapsto f_1$. We will show that $\pi_1 \circ \tilde{K}$ is holomorphic. The proof for the rigging at infinity is the same, except for the introduction of the map S . (The normalization causes no difficulties).

Let $t \mapsto [A, h_t, A_t]$ be a holomorphic curve in $T(A)$ for t in a neighbourhood $N \subset \mathbb{C}$ of zero. Since the fundamental projection $\Phi : L_{-1,1}^\infty(A) \rightarrow T(A)$ is holomorphic and has holomorphic sections, we can assume without loss of generality that $t \mapsto \mu(h_t)$ is holomorphic.

Note that \tilde{h} in (Definition 3.10) of K is related to h by a sewing operation and composition on the left by a holomorphic map. The normalization is also a left composition by a holomorphic map. So by the holomorphicity of the sewing operation [16, Lemma 6.3], $t \mapsto \mu(\tilde{h}_t)$ is a holomorphic map into $L_{-1,1}^\infty(\overline{\mathbb{C}})$. Because \tilde{h}_t is normalized, it is the unique solution to

the Beltrami equation with this normalization and with dilatation $\mu(\tilde{h}_t)$. Since solutions to the Beltrami equation depend holomorphically on μ [6, Proposition 4.7.6], we see that $\tilde{h}_t(z)$ is holomorphic in t for fixed z .

Denote $f_t = \tilde{h}_t \circ \tau_0$ and $g_t = \tilde{h}_t \circ \tau_\infty$. Then $f_t(z)$ is separately holomorphic in t and z , so it is jointly holomorphic. Thus all of the z -derivatives of $f_t(z)$ are holomorphic in t on \mathbb{D} .

Thus, $\mathcal{A}(f_t)(z)$ is holomorphic in t for all fixed z , and $f'_t(0)$ is holomorphic in t . Define $E_z : A_1^\infty \rightarrow \mathbb{C}$ to be the point evaluation functional $E_z(\rho) = \rho(z)$. These are continuous linear functionals for all $z \in \mathbb{D}$. For any open subset $D \subset \mathbb{D}$, $G = \{E_z : z \in D\}$ form a separating subset of the dual of A_1^∞ . By the previous paragraph, $E_z(\mathcal{A}(f_t)) = \mathcal{A}(f_t)(z)$ are holomorphic in t . So by [4] (see also [18, Theorem 3.8]), if we show that $t \mapsto (\mathcal{A}(f_t), f'_t(0))$ is locally bounded, we will have shown that \tilde{K} is Gâteaux holomorphic. In fact, if we show that \tilde{K} is locally bounded, we can further conclude that \tilde{K} is holomorphic (see for example [3, p 198] or [18, Theorem 3.7]).

We show that \tilde{K} is bounded. By [6, Theorem 4.7.4] applied to $\tilde{h} \circ \tau_0$ and $\tilde{h} \circ \tau_\infty$, there is a neighbourhood W of $[A, h, A']$ in $T(A)$ such that $\tilde{K}([A, h_1, A_1]) \subset V$ for all $[A, h_1, A_1] \in V$. Now if $(f, g) \in V$, we have that $f(\mathbb{D}) \subset U^0$. Since the closure of B^0 does not contain ∞ , we can assume that $B^0 \subset \{z : |z| < R\}$ for some $R > 0$. Thus $|f'(0)| \leq R$ by the Schwarz lemma. By the second coefficient estimate for univalent functions, we have that

$$|(1 - |z|^2)\Psi(f_1)(z) - 2\bar{z}| \leq 4$$

so

$$\|(1 - |z|^2)\Psi(f_1)(z)\|_\infty \leq 6$$

for all $f_1 \in \pi_1(V)$. A similar argument shows that $S(g)'(0)$ and $\Psi(S(g))$ are bounded. Thus \tilde{K} is bounded on W . This completes the proof. \square

Finally, we observe that multiplication of annuli is holomorphic.

Corollary 4.5. *Multiplication of annuli is holomorphic, both in the complex structure of non-overlapping maps, and in the complex structure inherited from the Teichmüller space of annuli.*

Proof. By Theorems 4.3 and 4.4, it is enough to show that multiplication is holomorphic in the complex structure inherited by the Teichmüller space of annuli. Thus the claim follows from 4.2. \square

It also follows immediately that

Corollary 4.6. *The complex structure on \mathcal{A}^0 inherited from $T(A)$ does not depend on the choice of standard base (A, τ_0, τ_∞) .*

Remark 4.7. This also follows from [16, Theorem 5.4].

4.3. Summary of mappings. For the convenience of the reader, we provide a diagram illustrating the maps identifying the various spaces. The bottom row is from Theorem 2.20.

(4.1)

$$\begin{array}{ccccccc}
& & T(A) & & & & \\
& & \downarrow P & \searrow K & & & \\
& & T(A)/\mathbb{Z} & & & & \\
& \swarrow F & & \swarrow K \circ P^{-1} & & & \\
\widetilde{\mathcal{M}}(0, 2) & \xrightarrow[\cong]{R} & \mathcal{A}^o & \xrightarrow[\cong]{I} & \mathcal{O}_{\text{qc}}(\mathbb{C}^*)/\text{Aut}(\mathbb{C}^*) & \xrightarrow[\cong]{r_a} & \mathcal{O}_{\text{qc}}(\mathbb{C}^*)_a \subset \mathcal{O}_{\text{qc}}(\mathbb{C}^*) \\
& & \swarrow L & & & & \\
& & & & & &
\end{array}$$

4.4. A complex structure on bounded univalent functions. The bounded univalent functions are a subsemigroup of the semigroup of rigged annuli [13, 19]. In this section, we show that the bounded univalent functions with quasiconformal extensions \mathcal{E}^0 inherits a complex structure from \mathcal{A}^0 in which composition is holomorphic.

Theorem 4.8. \mathcal{E}^o is a complex submanifold of \mathcal{A}^o .

Proof. By the definition of the complex structure on \mathcal{A}^o and Corollary 2.19, it suffices to show that \mathcal{E}^o is a complex submanifold of $\mathcal{O}_{\text{qc}}(\mathbb{C}^*)$. Fix a point $(f, \text{Id}) \in \mathcal{E}^o \subset \mathcal{A}^o$. We have a global embedding (Section 2.4)

$$B(f_1, g_1) = (\mathcal{A}(f_1), f_1'(0), \mathcal{A}(S(g_1)), g_1'(\infty)).$$

For every element $(f_1, g_1) \in \mathcal{E}^o$,

$$B(f_1, g_1) = (\mathcal{A}(f_1), f_1'(0), 0, 1).$$

It follows that \mathcal{E}^o is a complex submanifold of \mathcal{A}^0 . \square

Corollary 4.9. *The set of normalized quasiconformally extendible bounded univalent maps \mathcal{B} possesses a complex structure, in which composition is holomorphic. This complex structure is compatible with that inherited from the Teichmüller space of the annulus $T(A)$.*

Proof. By Corollary 4.5, multiplication is holomorphic. By Proposition 2.8, multiplication in \mathcal{E}^o is given by $(f_1, \text{Id}) \cdot (f_2, \text{Id}) = (f_1 \circ f_2, \text{Id})$. \square

This is an interesting application of the ideas of conformal field theory to geometric function theory. As far as we know, composition in the semigroup of bounded univalent functions has not been related to the Teichmüller space of doubly-connected domains and its complex structure.

Note that two fundamental semigroups in geometric function theory, quasiasymmetries and bounded univalent functions, both under composition, are in some sense “interpolated” by the Neretin-Segal semigroup. On the other hand, normalized quasiasymmetries are a model of the universal Teichmüller space, but composition of quasiasymmetries is not even continuous in the universal Teichmüller space. There is no contradiction, since \mathcal{G} is not a subset of \mathcal{A}^0 . It is of some interest to investigate whether \mathcal{G} is contained in an appropriately defined boundary of \mathcal{A}^0 (see [19, 444–445]).

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