A correspondence between Teichmüller theory and conformal field theory

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McGill analysis seminar

Partners and attributions

Most of this work is joint with David Radnell (American Univ. of Sharjah). Recent work is with Wolfgang Staubach (Uppsala Universitet).

It is part of a program to rigorously construct conformal field theory according to Segal, Kontsevich and others.

Partly "contract work" for Yi-Zhi Huang (Rutgers), one of the main actors in the program of constructing CFT from vertex operator algebras.

Partly we are exploiting the overlap between the fields to get new results in geometric function theory and Teichmüller theory.

Outline

Background in Teichmüller theory

1. Quasiconformal maps: deformations of Riemann surfaces
2. Definition of Teichmüller space

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1. Quasiconformal maps: deformations of Riemann surfaces
2. Definition of Teichmüller space

A connection between Teichmüller space and conformal field theory

4. A few words about conformal field theory
5. A correspondence between moduli spaces
6. An informal sketch of the consequences of the correpsondence.

Equivalence of Riemann surfaces

Two Riemann surfaces Σ_1 and Σ_2 are equivalent if:
there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$.

Classical problem: classify Riemann surfaces up to equivalence.

An example: When are annuli equivalent

Theorem

Two annuli $A_1 = \{z : r_1 < |z| < R_1\}$ *and* $A_2 = \{z : r_2 < |z| < R_2\}$ *are biholomorphic if and only if* $R_2/r_2 = R_1/r_1$.

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By rescaling, we can always choose $r = 1$.

An example: When are annuli equivalent

Proof.

(One direction only)

Let u_i be the harmonic function on A_i which is 1 on the outer boundary and 0 on the inner boundary. If $f : A_1 \to A_2$ is a biholomorphism then clearly $u_1 = u_2 \circ f$.

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Let u_i be the harmonic function on A_i which is 1 on the outer boundary and 0 on the inner boundary. If $f : A_1 \to A_2$ is a biholomorphism then clearly $u_1 = u_2 \circ f$.
Change of variables shows that

$$\iint_{A_2} |\nabla u_2|^2 dA = \iint_{A_1} |(\nabla u_2) \circ f|^2 |f'|^2 dA = \iint_{A_1} |\nabla u_1|^2 dA.$$

A bit of work shows that $u_i = (\log R_i/r_i)^{-1} \log(|z|/r_1)$ and that

$$\iint_{A_i} |\nabla u_i|^2 dA = \frac{2\pi}{\log \frac{R_i}{r_i}} \quad \Rightarrow \quad \frac{R_1}{r_1} = \frac{R_2}{r_2}.$$

□

Local deformations of Riemann surfaces

Idea: Rather than considering a set of Riemann surfaces, consider a set of maps from a fixed Riemann surface to inequivalent surfaces.

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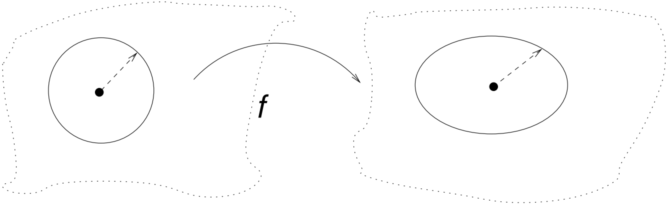
Teichmüller theory in a nutshell:

1. Teichmüller space is the set of local deformations of Riemann surfaces
2. Local deformations are quasiconformal maps.
3. It imposes a *weaker* equivalence than conformal equivalence. In general Teichmüller space is a cover of the moduli space of surfaces up to conformal equivalence.
4. Teichmüller space can be modelled by function spaces (in several ways); locally it is a Banach space.

Idea of quasiconformal map



A quasiconformal map is one such that the Jacobian matrix takes circles to ellipses of bounded distortion: i.e. ratio of the major to minor axes is bounded by a fixed constant.

Idea: angle is distorted.

Precise definition

Let $A, B \subset \mathbb{C}$ be open and connected.

Definition

A **quasiconformal map** $f : A \to B$ is an orientation-preserving homeomorphism such that

1. f is absolutely continuous on horizontal and vertical lines
2. There is a fixed constant $k < 1$ such that

$$\left| \frac{\partial f}{\partial \bar{z}} \right| \leq k \left| \frac{\partial f}{\partial z} \right|$$

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Quasiconformal maps have a *weaker local condition* but a *stronger global condition* than a C^∞ homeomorphism.

Examples

Affine map between rectangles:

$B_i = \{z = x + iy : 0 < x < r_i,\ 0 < y < 2\pi\}$

$$f(z) = az + b\bar{z} \qquad a = \frac{1}{2}\left(\frac{r_2}{r_1} + 1\right), b = \frac{1}{2}\left(\frac{r_2}{r_1} - 1\right).$$

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Map between annuli: $A_i = \{z : 1 < |z| < R_i\}$

$$f(z) = \exp\left(\alpha \log z + \beta \log \bar{z}\right)$$

for suitable choice of α and β. If $R_1 \neq R_2$, then $\beta \neq 0$ so this is quasiconformal but not conformal.

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Disc and plane
There is *no* quasiconformal map between $\{z : |z| < 1\}$ and $\mathbb{C}$, but there is a C^∞ homeomorphism.

Holomorphic deformations

Theorem (λ-lemma, Mañé, Sad, Sullivan; later Slodkowski 1991)

Let A be any subset of the plane. Let $f : N \times A \to \mathbb{C}$ be "holomorphic motion"; that is a map satisfying

1. *N is an open connected subset of $\mathbb{C}$ containing 0.*
2. *For any fixed $z \in A$, $\lambda \mapsto f(\lambda, z)$ is holomorphic.*
3. *For any fixed $\lambda \in N$, $z \mapsto f(\lambda, z)$ is one-to-one.*
4. *$f(0, z) = z$ for all $z \in A$.*

Then $f(\lambda, \cdot)$ has a quasiconformal extension to $\mathbb{C}$ for each λ, which is holomorphic in λ for each fixed z.

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Then $f(\lambda, \cdot)$ has a quasiconformal extension to $\mathbb{C}$ for each λ, which is holomorphic in λ for each fixed z.

So: *any* holomorphic perturbation of the identity map is quasiconformal.
So: any holomorphic deformation of a Riemann surface is quasiconformal.

Teichmüller space

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Fix a Riemann surface Σ. Its Teichmüller space $T(\Sigma)$ is

$$\{(\Sigma, f, \Sigma_1)\}/\sim$$

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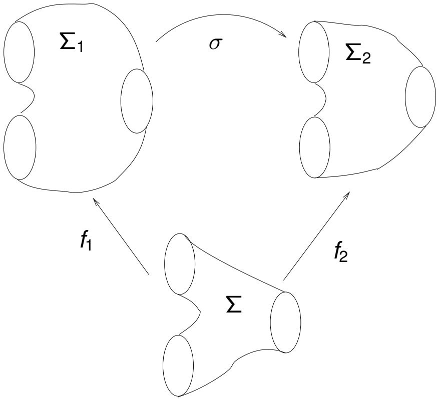
rel boundary means the homotopy is the identity map on the boundary of Σ.

Teichmüller equivalence

$f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity.



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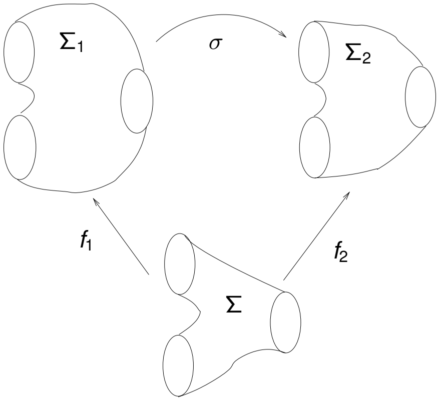

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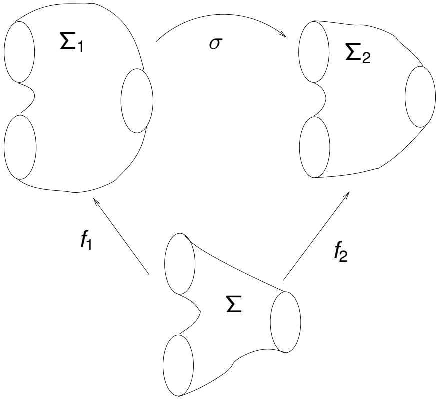

- Σ compact minus points $\Rightarrow T(\Sigma)$ finite dimensional.
- Σ has boundary curves $\Rightarrow T(\Sigma)$ infinite dimensional.

Complex structures on Teichmüller space

Theorem (Ahlfors, 1960)

Let R be a compact Riemann surface of genus g with n points removed. Assume that $2g - 2 + n > 0$. The Teichmüller space of R is a $3g - 3 + n$ dimensional complex manifold.

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More generally,

Theorem (Bers, 1964,1965)

The Teichmüller space of a Riemann surface is a complex Banach manifold.

Example: universal Teichmüller space

Let Σ be the disc $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$.
Let $\mu \in L^\infty(\mathbb{D}^*)$, $\|\mu\|_\infty = k < 1$. Let $f_\mu : \mathbb{D}^* \to \Sigma_1$ be the solution to the Beltrami differential equation

$$\frac{\overline{\partial} f}{\partial f} = \mu.$$

Let g_μ be the quasiconformal map satisfying

1. $\overline{\partial} g / \partial g = \mu$ on $\mathbb{D}^*$
2. $\overline{\partial} g / \partial g = 0$ on $\mathbb{D}$
3. $g(0) = 0$, $g'(0) = 1$, $g''(0) = 0$.

Teichmüller equivalence: $[\mathbb{D}^*, f_\mu, \Sigma_1] = [\mathbb{D}^*, f_\nu, \Sigma_2]$ if and only if $g_\mu = g_\nu$ on $\mathbb{D}$.

So $T(\mathbb{D}^*)$ is a function space of conformal maps $g : \mathbb{D} \to \mathbb{C}$.

Example continued

Let $S(g)$ denote the Schwarzian derivative

$$S(g) = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right).$$

Theorem (Bers, classical)

The set of Schwarzian derivatives of g arising from the Teichmüller space above is an open subset of the Banach space of holomorphic functions

$$\{h : \mathbb{D} \to \mathbb{C} \,:\, \|(1 - |z|^2)^2 h(z)\|_\infty < \infty\}.$$

What is conformal field theory?

Conformal Field Theory (CFT) is:

- Special class of quantum/statistical field theories, invariant under local rescaling and rotation.
- Mathematical definition (G. Segal, Kontsevich $\approx$ 1986)
- Requires results in algebra, topology and analysis.
- Concerned with 2D CFT.

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Our General Aim:

- Provide a natural analytic setting for the rigorous definition of CFT in higher genus.
- Use CFT ideas to prove new results in Teichmüller theory and geometric function theory.

Rigged moduli space of Friedan and Shenker

Another kind of moduli space appears in CFT.

Let Σ^P be a compact Riemann surface of genus g with n (ordered) punctures.

Definition (Riggings)

A "rigging" is an n-tuple of maps $\phi = (\phi_1, \ldots, \phi_n)$ where $\phi_i : \mathbb{D} \to \Sigma^P$ is a one-to-one, holomorphic map of the unit disc $\mathbb{D}$, taking 0 to the puncture, satisfying

$$\overline{\phi_i(\mathbb{D})} \cap \overline{\phi_j(\mathbb{D})} = \emptyset$$

whenever $i \neq j$.

Riggings: important analytic point

Technical but important point: David and I assume that each map ϕ_i has a quasiconformal extension to a neighbourhood of $\overline{\mathbb{D}}$.

In CFT, it is customary to assume that ϕ_i extend diffeomorphically or analytically to the closure of $\mathbb{D}$. Our definition is more general:

$$\begin{aligned} \text{Diff extble. to } (\overline{\mathbb{D}}) \quad &\subsetneq \quad \text{QC-extble to } (\overline{\mathbb{D}}) \\ &\subsetneq \quad \text{Homeomorphically extble. to } (\overline{\mathbb{D}}). \end{aligned}$$

In our opinion, the question of what choice of riggings is central (in particular, leads to connection with Teichmüller theory).

Rigged moduli space of Friedan and Shenker

Definition (Rigged moduli space)

The "rigged Riemann moduli space" of type (g, n) is

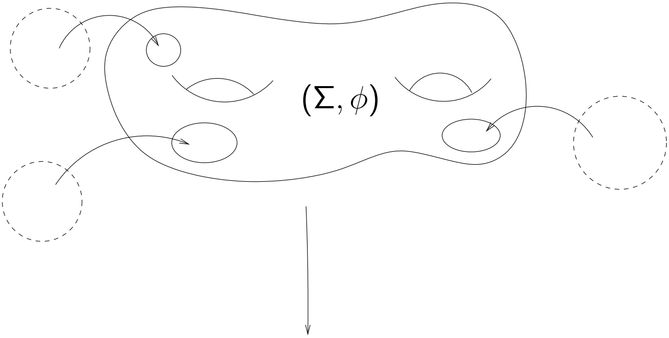
$$\widetilde{\mathcal{M}}(g, n) = \{(\Sigma, \phi)\}/\sim$$

where

- Σ is a compact genus g Riemann surface, with n punctures
- ϕ is a rigging
- $(\Sigma_1, \phi_1) \sim (\Sigma_2, \phi_2)$ if there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $\phi_2 = \sigma \circ \phi_1$.

Needs of conformal field theory I

Let $\mathcal{H}$ be a Hilbert space.



$$A(\Sigma, \phi) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$$

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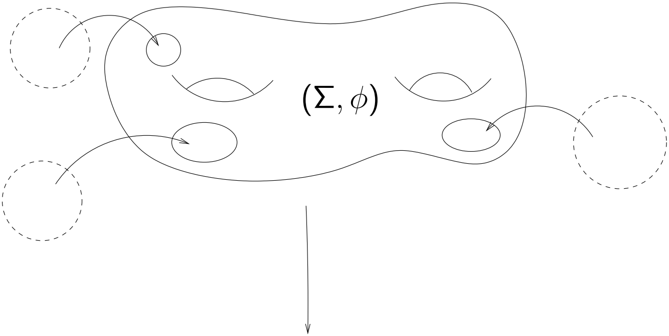

$$A(\Sigma, \phi) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$$

It is required that $A(\Sigma, \phi)$ depends holomorphically on (Σ, ϕ).

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Focus on first two problems in the remainder of the talk.

Rigged moduli space is almost Teichmüller space

Theorem (Radnell and S, 2006 Commun. Contemp. Math.)

Let Σ^B be a bordered Riemann surface, biholomorphic to a compact surface of genus g with n discs removed.

$$\widetilde{\mathcal{M}}(g,n) = T(\Sigma^B)/G.$$

The action by G is fixed-point-free and properly discontinuous.

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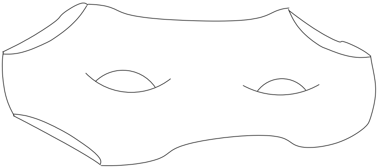
Proof.

- Extension theorems for quasiconformal maps, quasisymmetries, etc.
- Lambda-lemma of Mañe, Sad and Sullivan.

□

Correspondence between moduli spaces

Base surfaces:



Σ^B

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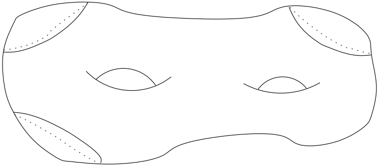

$$\Sigma^B \subset \Sigma^P$$

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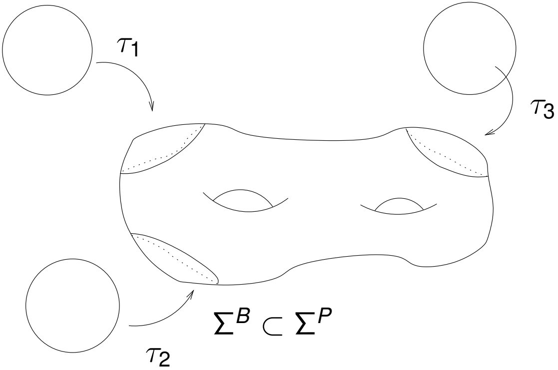
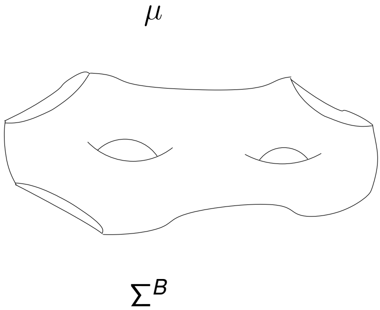


Correspondence between moduli spaces continued

Map from $T(\Sigma^B)$ to $\widetilde{\mathcal{M}}(g,n)$:
Let μ be a Beltrami differential



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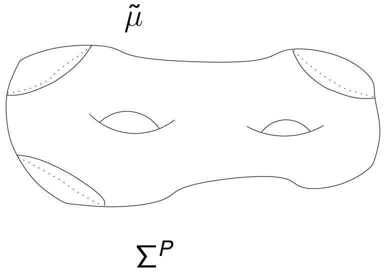

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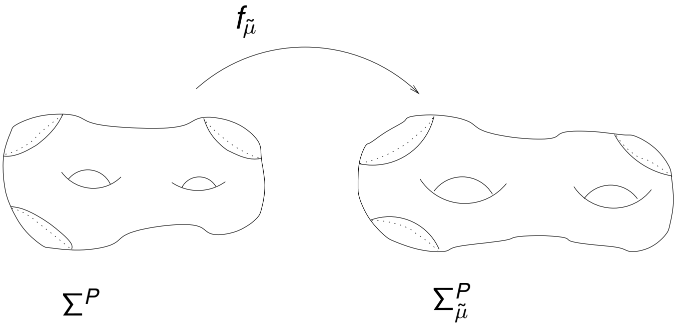

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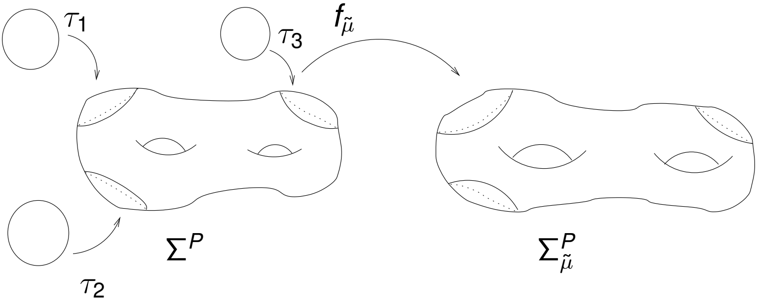

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What is the correspondence good for?

Theorem (Radnell and S, 2006 Commun. Contemp. Math.)

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The G-action is fixed-point-free and properly discontinuous.

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Question one. What can Teichmüller theory do for conformal field theory?

Question two. What can conformal field theory do for Teichmüller theory?

Consequences for conformal field theory

Theorem (Radnell and S, 2006 Commun. Contemp. Math.)

The rigged moduli space $\widetilde{\mathcal{M}}(g,n)$ is a Banach manifold. The operation of sewing is holomorphic.

Consequences for Teichmüller theory: geometric structure

Fiber structure of Teichmüller space [Radnell & S, Journal d'Analyse 2009, Conf. Geom. and Dynamics 2010] Let Σ^P be obtained from a bordered surface Σ^B by sewing on punctured discs.

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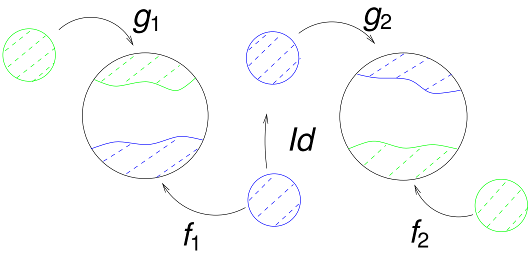
Application: new explicit coordinates on Teichmüller space.

Consequences for Teichmüller theory continued: algebraic structure

The set of pairs of non-overlapping maps is a semi-group, which is in some sense the exponential of the Witt algebra (Neretin, Segal).

Sew with blue riggings:



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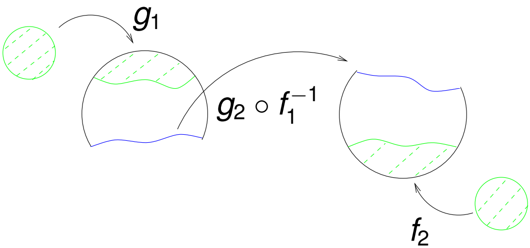

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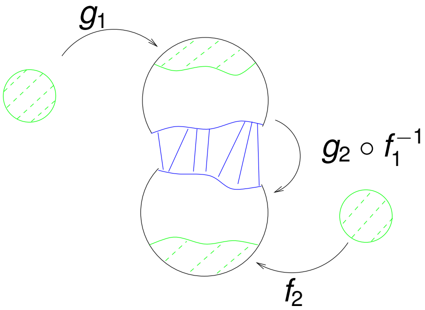

Consequences for Teichmüller theory continued: algebraic structure

Theorem (Radnell and S, J. Lond. Math. Soc 2012)

The Neretin-Segal semigroup is $T(A)/\mathbb{Z}$ *where* A *is an annulus. The Neretin-Segal semigroup is a complex Banach manifold in which multiplication is holomorphic.*

Corollary

$T(A)/\mathbb{Z}$ *possesses a holomorphic multiplication.*

More consequences and future work

Teichmüller space is a Hilbert manifold. [Radnell, S, Staubach]

- There is a natural refinement of Teichmüller space of a bordered surface possessing a Hilbert manifold structure.
- This refinement is holomorphically included in the Teichmüller space.

Partly motivated by work of Takhtajan and Teo (Memoirs of the AMS, 2006)

Future work [Radnell, S, Staubach]:

- Construct determinant line bundle of $\overline{\partial} \oplus \mathrm{pr}$ over the rigged moduli space
- Relate to the Weil-Petersson metric (natural metric in Teichmüller theory).

The End

Thanks!