# The Weil-Petersson metric on a Teichmüller space of bordered surfaces

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# Joint with

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# Blanket assumptions

In this talk we consider Riemann surfaces $\Sigma$ with:

1. $g$ handles, $g \geq 0$
2. a border (in the sense of e.g. Ahlfors and Sario)
3. $n$ boundary components, each homeomorphic to $\mathbb{S}^1$

Call this "type $(g, n)$ bordered Riemann surface".

we also assume

1. $2g - 2 + n > 0$
2. $\Sigma$ has a hyperbolic metric; i.e. covered by the disc.

**Notation** $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{D}^* = \{|z| > 1\} \cup \{\infty\}$

# Definition of Teichmüller space

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**rel boundary** means the homotopy is the identity map on the boundary of $\Sigma$.

# Differentials

A $(k, l)$-differential given in local coordinates $z$ by $\alpha(z)$ transforms under biholomorphic change of parameter $z = g(w)$ according to

$$\alpha(z) = \tilde{\alpha}(g(w))g'(w)^k\overline{g'(w)}^l.$$

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Useful fact: you may multiply and divide differentials.

$$\alpha(z)dz^k d\bar{z}^l \; \times \; \beta(z)dz^m d\bar{z}^l \rightarrow \alpha(z)\beta(z)dz^{k+m}d\bar{z}^{l+n}.$$

The right hand side transforms correctly!
We will write $\alpha \cdot \beta$ for short.

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**Example 4** A Beltrami differential is a $(-1, 1)$-differential.

**Example 5**

$$\Omega_{-1,1}(\Sigma) = \{\lambda_\Sigma^{-2}\overline{\alpha} : \alpha \text{ holo } (2,0) - \text{diff and } \|\lambda_\Sigma^{-2}\overline{\alpha}\|_\infty < \infty\}$$

# Tangent space to Teichmüller space

Let $\Phi$ be the map into the solution to the Beltrami equation:

$$\Phi : L^{\infty}_{-1,1}(\Sigma)_1 \to T(\Sigma)$$
$$\mu \mapsto [\Sigma, f, \Sigma_1] \qquad \text{where } \overline{\partial} f = \mu \partial f.$$

$L^p_{-1,1}(\Sigma)$ is the space of $L^p$ Beltrami differentials (with respect to hyperbolic area measure).
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**Classical facts**:

1. $L_{-1,1}(\Sigma) = \ker D\Phi_{id} \oplus \Omega_{-1,1}(\Sigma)$.
2. $\Phi$ is holomorphic.
3. $\Phi|_{\Omega_{-1,1}(\Sigma)}$ is biholomorphic on a small enough ball.

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So $\Omega_{-1,1}(\Sigma)$ is the tangent space at $id = [\Sigma, \mathrm{id}, \Sigma]$.

# Weil-Petersson

For $\phi, \psi \in \Omega_{-1,1}(\Sigma)$ we define the Weil-Petersson pairing

$$\langle \phi, \psi \rangle = \iint_{\Sigma} \phi \overline{\psi} dA_{\Sigma}$$

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But why should it converge? It converges in the compact case... what else?

**Example**: $T(\mathbb{D})$ is the universal Teichmüller space; can be modelled by $QS(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1)$ ($QS(\mathbb{S}^1) =$ quasisymmetries of circle).
Nag and Verjovsky showed that for $T(\mathbb{D})$ (the universal Teichmüller space), the Weil-Petersson metric does not converge; only for smooth quasisymmetries.

# Who cares about the Weil-Petersson metric?

Why is it interesting?

1. geometry of Teichmüller space described by this metric (e.g. Wolpert, relation to geodesic length functions; Masur and Wolf, isometries of WP metric are the mapping class group)
2. relates to spectral theory of the Laplacian (e.g. Wolpert, Takhtajan and Zograf, Takhtajan and Teo)
3. and index theorems for families of $\bar{\partial}$ operators and Quillen determinant line bundle (e.g. Wolpert, Biswas and Schumacher, Takhtajan and Zograf)

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Key point: does not converge on general Teichmüller spaces.
Our interest: determinant line bundle of certain elliptic operators and CFT, index theorems in the 2D conformal setting.

# Some literature

Recall: every element $\gamma \in QS(\mathbb{S}^1)$ can be represented as $g^{-1} \circ f$ for univalent quasiconformally extendible maps $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \mathbb{C}$.

## Definition

$$T_{WP}(\mathbb{D}) = \left\{ [\gamma] \; : \; \iint_{\mathbb{D}} \left| \frac{f''}{f'} \right|^2 < \infty \right\}.$$

Call such a quasisymmetric map a **"Weil-Petersson class" quasisymmetry.**

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- Guizhen Cui showed that every quasisymmetry has a quasiconformal extension to $\mathbb{D}$ such that Beltrami differential is in $L^2$. Defined "Weil-Petersson class" Teichmüller space above (w different terminology). Weil-Petersson metric converges.
- Guo Hui extended some of these results to $L^p$.

# Literature continued

- Leon Takhtajan and Lee-Peng Teo independently gave the same definition and also showed that
  - $T_{WP}(\mathbb{D})$ is a topological group (a Hilbert manifold)
  - gave potentials for WP-metric in terms of Grunsky operator
  - sewing formulas for Laplacian, etc. etc.
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Advice to young function theorists: read Takhtajan and Teo.

# Weil-Petersson class quasisymmetries

Boundary curves: $\partial_i \Sigma$ $i = 1, \ldots, n$.
A **collar chart** is a biholomorphism $H : A_i \to \{1 < |z| < r\}$ where $A_i$ is a doubly-connected neighbourhood of $\partial_i \Sigma$.

## Definition

Let $\Sigma$ and $\Sigma_1$ of type $(g, n)$ be Riemann surfaces, and $\gamma$ and $\gamma_1$ boundary curves. $\phi : \gamma \to \gamma_1$ is a **Weil-Petersson class quasisymmetry** if $H \circ \phi \circ H_1^{-1}$ is a WP-class quasisymmetry for collar charts $H$ and $H_1$ of $\gamma$ and $\gamma_1$.

# Weil-Petersson class Teichmüller space

Definition

Let $\Sigma$ be a Riemann surface of type $(g, n)$. The Weil-Petersson class Teichmüller space is

$$T_{WP}(\Sigma) = \{(\Sigma, f, \Sigma_1) \; : \; f|_{\partial_i \Sigma} \text{ WP-class quasisymmetry for all i}\} / \sim .$$

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### Theorem (Radnell, S, Staubach 1 (submitted))

*$T_{WP}(\Sigma)$ is a second countable, Hausdorff topological space, with an atlas of charts making it a Hilbert manifold.*

### Theorem (Radnell, S, Staubach 1 (submitted))

*The inclusion map $T_{WP}(\Sigma) \hookrightarrow T(\Sigma)$ is holomorphic.*

Long painful story for another day.

# Representatives

**Theorem (Radnell, S, Staubach 2 (submitted))**

*Let $\Sigma$ and $\Sigma_1$ be bordered Riemann surfaces of type $(g, n)$. Let $f : \Sigma \to \Sigma_1$ be such that the boundary values are WP-class quasisymmetries. There exists a quasiconformal $\hat{f} : \Sigma \to \Sigma_1$ such that*

1. *$\hat{f}$ is homotopic to $f$ rel boundary (so $[\Sigma, f, \Sigma_1] = [\Sigma, \hat{f}, \Sigma_1]$)*
2. *$\hat{f}$ has $L^2$ Beltrami differential.*

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2. *$\hat{f}$ has $L^2$ Beltrami differential.*

This generalizes the theorem of Guo Hui and Guizhen Cui. In fact

**Theorem (Radnell, S, Staubach 3 (submitted))**

*Let $[\Sigma, f_t, \Sigma_t]$ be a holomorphic curve through Id in $T_{WP}(\Sigma)$. For small $t$ representatives can be chosen so that $f_t$ has Beltrami differential in $L^2(\Sigma)$ and $t \mapsto \mu(f_t)$ is holomorphic both in $L^2_{-1,1}(\Sigma)$ and $L^\infty_{-1,1}(\Sigma)$.*

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Why isn't all this easy? $\Sigma$ is covered by $\mathbb{D}$. Just lift everything to the cover, and use the results of Hui, Cui, Takhtajan and Teo etc...

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Nope! For a fundamental domain $F \subset \mathbb{D}$, if a Beltrami differential has non-zero $L^2$ norm on $F \cong \Sigma$, then it has infinite $L^2$ norm on $\mathbb{D}$.

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So there is quite a bit of work involved. Involves our CFT/Teichmüller space correspondence in an essential way.

# Tangent space

Now $L^2(\Sigma) \cap L^{\infty}_{-1,1}(\Sigma)_1$ is not a particularly nice space (this problem already exists for $T_{WP}(\mathbb{D})$ - i.e. we didn't create it).

Definition

$$H_{-1,1}(\Sigma) = \left\{ \beta = \lambda_{\Sigma}^{-2}\overline{\alpha} \,:\, \alpha \text{ holo } (2,0)-\text{diff and } \iint_{\Sigma} |\beta|^2 \, dA_{\Sigma} < \infty \right\}$$

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## Theorem (Radnell, S, Staubach 3 (submitted))

*Let* $\mathbf{v}$ *be a tangent vector at the identity* $[\Sigma, id, \Sigma]$ *of* $T_{WP}(\Sigma)$*. There is a holomorphic curve* $[\Sigma, f_t, \Sigma_t]$ *in* $T_{WP}(\Sigma)$ *tangent to* $\mathbf{v}$ *at* $t = 0$*.*

# Model of tangent space

Recall $\Phi$ takes a Beltrami differential $\mu$ to the Teichmüller space element $[\Sigma, f, \Sigma_1]$ (where $\bar{\partial} f = \mu \partial f$).

Theorem (Radnell, S, Staubach 3 (submitted))

$$L^{\infty}_{-1,1}(\Sigma) \cap L^{2}_{-1,1}(\Sigma) = ker\, D\Phi_{id}|_{L^2_{-1,1}(\Sigma)} \oplus H_{-1,1}(\Sigma).$$

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**Theorem (Radnell, S, Staubach 3 (submitted))**

*For a small enough ball* $0 \in B \subset H_{-1,1}(\Sigma)$*,* $\Phi : B \to T_{WP}(\Sigma)$ *is a biholomorphism onto its image.*

In fact, you can get an atlas of charts this way using the change of base point map.

# Convergence of Weil-Petersson metric

In summary: $H_{-1,1}(\Sigma)$ is the tangent space at *id* of $T_{WP}(\Sigma)$.

In the case of $\Sigma = \mathbb{D}$ this is due to Takhtajan and Teo.

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Corollary (Radnell, S, Staubach 3 (submitted))

*The Weil-Petersson metric converges on $T_{WP}(\Sigma)$.*

**Remark**: away from the identity you can describe the tangent space using the change of base point map. Left this out to simplify the talk.

# The end!!

Thanks!

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