WEIL-PETERSSON CLASS NON-OVERLAPPING MAPPINGS INTO A RIEMANN SURFACE

DAVID RADNELL, ERIC SCHIPPERS, AND WOLFGANG STAUBACH

Abstract. For a Riemann surface of genus g with n punctures, consider the class of n-tuples of conformal mappings $(\phi_1, \ldots, \phi_n)$ each taking 0 to a puncture. Assume further that the pre-Schwarzians of each ϕ_i is in the Bergman space and the extensions to the closure of the disk do not intersect. We show that the class of such non-overlapping mappings is a complex Hilbert manifold.

1. Introduction

Let Σ be a Riemann surface of genus g with n ordered punctures p_i. We consider the set of n-tuples of conformal maps $(\phi_1, \ldots, \phi_n)$ from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ into Σ, such that $\phi_i(0) = p_i$ for each $i = 1, \ldots, n$. We assume that these maps have quasiconformal extensions to an open neighbourhood to the closure of $\mathbb{D}$, and have pre-Schwarzians in the Bergman space. We furthermore assume that the closures of the images do not overlap. We will refer to such an n-tuple of as a Weil-Petersson class rigging and denote the set of such mappings by $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$. We show that the set of such mappings is a complex Hilbert manifold.

There are two motivations for this result. The first comes from Teichmüller theory. The results of this paper can be used to construct a quasiconformal Teichmüller space of genus g surfaces bordered by n boundary curves, which is a complex Hilbert manifold. This "Weil-Petersson class Teichmüller space" is holomorphically immersed into the standard quasiconformal Teichmüller space. It turns out that this result will enable us to show that the Weil-Petersson class Teichmüller space possesses a convergent Weil-Petersson metric, although this step requires substantial work and is dealt with in a forthcoming publication. It is worth mentioning that the construction of a Weil-Petersson metric will be the first instance of the existence of such metrics in the infinite dimensional setting for bordered Riemann surfaces of genus $g > 0$ or $n > 1$. This construction can not be achieved without solving the analytic problems settled in this paper. Indeed, one of the main technical difficulties in this paper is to show that the transition functions of the atlas defining the Hilbert manifold structure on the space of riggings are biholomorphisms. This is a consequence of some analytic problems that are of independent interest in geometric function theory and the theory of quasiconformal mappings. To solve these, we require (among other things) the theory of Carleson measures for analytic Besov spaces and also utilize the relationship between the Dirichlet space and the little Bloch space.

The second motivation comes from two-dimensional conformal field theory. In [17], two of the authors showed that a moduli space of so-called "rigged Riemann surfaces" (due to D. Friedan and S. Shenker [8] G. Segal [21] and C. Vafa [25]) is the quotient of the standard

---

1991 *Mathematics Subject Classification.* Primary 30F60 ; Secondary 30C55, 30C62, 32G15, 46E20, 81T40.

Eric Schippers is partially supported by the National Sciences and Engineering Research Council. He would like to thank Nina Zorboska for several helpful conversations.

quasiconformal Teichmüller space of bordered surfaces by a discrete group. This allowed us to construct a complex structure on the rigged moduli space, and show that the operation of sewing is holomorphic, which are necessary steps in the rigorous construction of conformal field theory. However, it appears that the standard Banach manifold structure of the standard Teichmüller space is not regular enough for the construction of the determinant line bundle of Cauchy-Riemann operators with boundary projection factor. The Weil-Petersson class Teichmüller space mentioned above appears to have sufficient regularity (preliminary results in this direction can be found in [20]).

In [19], two of the authors demonstrated that the standard Teichmüller space of a genus g surface with n boundary curves homoeomorphic to $\mathbb{S}^1$ is holomorphically fibered over the Teichmüller space of surfaces of genus g with n punctures. The fibers are exactly the set of non-overlapping maps into a surface, which we called riggings above. However in that case a given rigging $(\phi_1, \ldots, \phi_n)$ has weaker regularity; namely $\log f'$ is in the Bloch space. By establishing in this paper that the Weil-Petersson class riggings (with pre-Schwarzian in the Bergman space) is a complex Hilbert manifold, we have thus endowed the fibers of the Weil-Petersson class Teichmüller space with this complex structure. In order to construct the complex structure, one would need to combine the results of [19] and those obtained here regarding Weil-Petersson class Teichmüller space.

We now fill out this explanation of the motivation for the results of this paper with a more detailed discussion of the literature. There have been several refinements of quasiconformal Teichmüller space, obtained by considering natural analytic subclasses either of the quasisymmetries of the circle or of the quasiconformally extendible univalent functions in the Bers model of universal Teichmüller space. For example, K. Astala and M. Zinsmeister [2] gave a model of the universal Teichmüller space based on BMO, and G. Cui and M. Zinsmeister [5] studied the Teichmüller spaces compatible with Fuchsian groups in this model. F. Gardiner and D. Sullivan [10] studied a refined class of quasisymmetric mappings (which they call symmetric) and the topology of this refined class.

A family of refined models of the universal Teichmüller space was given by G. Hui [14], each based on an L^p norm. These spaces were completely characterized in three ways: in terms of a space of quadratic differentials, in terms of univalent functions, and in terms of a space of Beltrami differentials; all satisfying a weighted L^p-type integrability condition. In this paper, we are concerned with the L^2 case, which G. Hui attributes to Cui [4]. In this case Y. Shen [22] also gave a fourth characterization by determining the precise analytic class of associated quasisymmetric mappings of the circle. Independently of Hui and Cui, L. Takhtajan and L.-P. Teo [23] defined a Hilbert manifold structure on the universal Teichmüller space and universal Teichmüller curve, equivalent to that of Hui, and obtained further far-reaching geometric and analytic results; for example they gave explicit forms for the Kähler potential of the Weil-Petersson metric. Since then there has been growing interest in Weil-Petersson class mappings and quasisymmetries; for a brief survey see the introduction to Shen [22].

2. WP-class conformal maps

In Section 2 we collect some known results on the refinement of the set of quasisymmetries and quasiconformal maps, from the work of Takhtajan and Teo [23], Teo [24] and Hui [14]. We also derive two technical lemmas which follow from previous work of two of the authors [18]. We collect some necessary results on the Weil-Petersson class universal Teichmüller space of Takhtajan and Teo [23] and Hui [14]. We need to consider a smaller class than the class of quasisymmetric mappings of $\mathbb{S}^1$; we will refer to elements of this smaller class as WP-class quasisymmetries.

In [18] we defined the set $\mathcal{O}^{\mathrm{qc}}$ of quasiconformally extendible conformal maps of $\mathbb{D}$ in the following way.

Definition 2.1. Let $\mathcal{O}^{\mathrm{qc}}$ be the set of maps $f : \mathbb{D} \to \mathbb{C}$ such that f is one-to-one, holomorphic, has quasiconformal extension to $\mathbb{C}$, and $f(0) = 0$.

A Banach space structure can be introduced on $\mathcal{O}^{\mathrm{qc}}$ as follows. Let

$$A_1^\infty(\mathbb{D}) = \left\{ \phi \in \mathcal{H}(\mathbb{D}) : \|\phi\|_{A_1^\infty(\mathbb{D})} = \sup_{z\in\mathbb{D}} (1-|z|^2)|\phi(z)| < \infty \right\}. \tag{2.1}$$

This is a Banach space. It follows directly from results of Teo [24] that for

$$\mathcal{A}(f) = \frac{f''}{f'}$$

the map

$$\begin{aligned} \chi : \mathcal{O}^{\mathrm{qc}} &\to A_1^\infty(\mathbb{D}) \oplus \mathbb{C} \\ f &\mapsto (\mathcal{A}(f), f'(0)) \end{aligned} \tag{2.2}$$

takes $\mathcal{O}^{\mathrm{qc}}$ onto an open subset of the Banach space $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$ (see [18]). Thus $\mathcal{O}^{\mathrm{qc}}$ inherits a complex structure from $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$. The space $\mathcal{O}^{\mathrm{qc}}$ can be thought of as a two complex dimensional extension of the universal Teichmüller space.

We will construct a Hilbert structure on a subset of $\mathcal{O}^{\mathrm{qc}}$. To do this, in place of $A_1^\infty(\mathbb{D})$ we use the Bergman space

$$A_1^2(\mathbb{D}) = \left\{ \phi \in \mathcal{H}(\mathbb{D}) : \|\phi\|_2^2 = \iint_{\mathbb{D}} |\phi|^2 \, dA < \infty \right\}$$

which is a Hilbert space and a vector subspace of the Banach space $A_1^\infty(\mathbb{D})$. Furthermore, the inclusion map from $A_1^2(\mathbb{D})$ to $A_1^\infty(\mathbb{D})$ is bounded [23, Chapter II Lemma 1.3].

Here and in the rest of the paper we shall denote the Bergman space norm $\|\cdot\|_{A_1^2}$ by $\|\cdot\|$.

We define the class of WP-class quasiconformally extendible maps of the $\mathbb{D}$ as follows.

Definition 2.2. Let

$$\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} = \left\{ f \in \mathcal{O}^{\mathrm{qc}} : \mathcal{A}(f) \in A_1^2(\mathbb{D}) \right\}.$$

We will embed $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ in the Hilbert space direct sum $\mathcal{W} = A_1^2(\mathbb{D}) \oplus \mathbb{C}$. Since $\chi(\mathcal{O}^{\mathrm{qc}})$ is open, $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}) = \chi(\mathcal{O}^{\mathrm{qc}}) \cap A_1^2(\mathbb{D})$ is also open, and thus $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ trivially inherits a Hilbert manifold structure from $\mathcal{W}$. We summarize this with the following theorem.

Theorem 2.3. *The inclusion map from $A_1^2(\mathbb{D}) \to A_1^\infty(\mathbb{D})$ is continuous. Furthermore $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})$ is an open subset of the vector subspace $\mathcal{W} = A_1^2(\mathbb{D}) \oplus \mathbb{C}$ of $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$, and the inclusion map from $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})$ to $\chi(\mathcal{O}^{\mathrm{qc}})$ is holomorphic. Thus the inclusion map $\iota : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \to \mathcal{O}^{\mathrm{qc}}$ is holomorphic.*

Remark 2.4. Although the inclusion map is continuous, the topology of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ is not the relative topology inherited from $\mathcal{O}^{\mathrm{qc}}$. It's enough to show that $A_1^2(\mathbb{D})$ does not have the relative topology from $A_1^\infty(\mathbb{D})$. To see this observe that if

$$f_t = \frac{1}{\sqrt{|\log(1-t)|}(1 - t^2 z^2)}$$

for $t < 1$, then as $t \to 1$ $\|f_t\| \to 0$ in $A^2_1(\mathbb{D})$ whereas $\|f_t\|_{A^\infty_1(\mathbb{D})} \to \pi/2$.

Lemma 2.5. *Let $f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Let h be a one-to-one holomorphic map defined on an open set W containing $\overline{f(\mathbb{D})}$. Then $h \circ f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Furthermore, there is an open neighborhood U of f in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ and a constant C such that $\|\mathcal{A}(h \circ g)\| \leq C$ for all $g \in U$.*

Proof. The map $h \circ f$ has a quasiconformal extension to $\mathbb{C}$ if and only if it has a quasiconformal extension to an open neighborhood of $\overline{\mathbb{D}}$ (although not necessarily with the same dilatation constant). Clearly $h \circ f$ has a quasiconformal extension to W, namely h composed with the extension of f. Thus $h \circ f$ has an extension to the plane, and so $h \circ f \in \mathcal{O}^{\mathrm{qc}}$.

We need only show that $\mathcal{A}(h \circ f) \in A^2_1(\mathbb{D})$. This follows from Minkowski's inequality:

$$\begin{aligned} \left(\iint_{\mathbb{D}} |\mathcal{A}(h \circ f)|^2 dA\right)^{1/2} &\leq \left(\iint_{\mathbb{D}} |\mathcal{A}(h) \circ f \cdot f'|^2 dA\right)^{1/2} + \left(\iint_{\mathbb{D}} |\mathcal{A}(f)|^2 dA\right)^{1/2} \\ &= \left(\iint_{f(\mathbb{D})} |\mathcal{A}(h)|^2 dA\right)^{1/2} + \left(\iint_{\mathbb{D}} |\mathcal{A}(f)|^2 dA\right)^{1/2} \end{aligned} \tag{2.3}$$

The first term on the right hand side is finite because h is holomorphic and $h' \neq 0$ on an open set containing $\overline{f(\mathbb{D})}$ so $\mathcal{A}(h)$ is bounded on $f(\mathbb{D})$. The second term is bounded because $f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. This proves the first claim.

To prove the second claim, observe that there is a compact set K contained in W which contains $\overline{f(\mathbb{D})}$ in its interior. By [18, Corollary 3.5] there is an open set $\hat{U}$ in $\mathcal{O}^{\mathrm{qc}}$ such that $\overline{g(\mathbb{D})}$ is contained in the interior of K for all $g \in \hat{U}$. Since the inclusion $\iota : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \to \mathcal{O}^{\mathrm{qc}}$ is continuous, we obtain an open set $\iota^{-1}(\hat{U}) \subset \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ with the same property. Let U be an open ball in $\iota^{-1}(\hat{U})$ containing f. There is a constant C_1 such that for any $g \in U$

$$\iint_{\mathbb{D}} |\mathcal{A}(g)|^2 \, dA \leq C_1$$

and a constant C_2 such that

$$\iint_{g(\mathbb{D})} |\mathcal{A}(h)|^2 \, dA \leq \iint_K |\mathcal{A}(h)|^2 \, dA \leq C_2.$$

Applying (2.3) completes the proof. □

We will also need a technical lemma on a certain kind of holomorphicity of left composition in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

Lemma 2.6. *Let E be an open subset of $\mathbb{C}$ containing 0 and Δ an open subset of $\mathbb{C}$. Let $\mathcal{H} : \Delta \times E \to \mathbb{C}$ be a map which is holomorphic in both variables and injective in the second variable and let $h_\epsilon(z) = \mathcal{H}(\epsilon, z)$. Let $\psi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ satisfy $\overline{\psi(\mathbb{D})} \subseteq E$. Then the map $Q : \Delta \mapsto \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ defined by $Q(\epsilon) = h_\epsilon \circ \psi$ is holomorphic in ϵ.*

Proof. We need to show that for fixed ψ, $\mathcal{A}(h_\epsilon \circ \psi)$ and $(h_\epsilon \circ \psi)'(0)$ are holomorphic in ϵ. First observe that all the z-derivatives of h_ϵ are holomorphic in ϵ for fixed z. Thus the second claim is immediate.

To prove holomorphicity of $\epsilon \mapsto \mathcal{A}(h_\epsilon \circ \psi)$, it is enough to show weak holomorphicity and local boundedness [12]; that is, to show local boundedness and that for some set of separating continuous functionals $\{\alpha\}$ in the dual of the Bergman space, $\alpha \circ \mathcal{A}(h_\epsilon \circ \psi)$ is holomorphic for all α. Let E_z be the point evaluation function $E_z\psi = \psi(z)$. These are continuous on the Bergman space and obviously separating on any open set. Since

$$\mathcal{A}(h_\epsilon \circ \psi) = \mathcal{A}(h_\epsilon) \circ \psi \cdot \psi' + \mathcal{A}(\psi)$$

clearly $E_z(\mathcal{A}(h_\epsilon \circ f))$ is holomorphic in ϵ.

So we only need to prove that $\mathcal{A}(h_\epsilon \circ \psi)$ and $(h_\epsilon \circ \psi)'(0)$ are locally bounded. The second claim is obvious. As above, by Minkowski's inequality (2.3) and a change of variables

$$\left(\iint_{\mathbb{D}} |\mathcal{A}(h_\epsilon \circ \psi)|^2 dA\right)^{1/2} \leq \left(\iint_{\psi(\mathbb{D})} |\mathcal{A}(h_\epsilon)|^2 dA\right)^{1/2} + \left(\iint_{\mathbb{D}} |\mathcal{A}(\psi)|^2 dA\right)^{1/2}.$$

Since $\mathcal{A}(h_\epsilon)$ is jointly holomorphic in ϵ and z and $\overline{\psi(\mathbb{D})} \subseteq E$ for any fixed ϵ_0, there is a compact set D containing ϵ_0 such that $|\mathcal{A}(h_\epsilon)|$ is bounded on $\psi(\mathbb{D})$ by a constant independent of $\epsilon \in D$. Since $\mathcal{A}(\psi)$ is in the Bergman space this proves the claim. □

3. Function-theoretic results on non-overlapping mappings

Let Σ be a genus g Riemann surface with n punctures. In this section we define the class of non-overlapping mappings $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$. We also establish some technical theorems which are central to the proof that it is a Hilbert manifold in Section 4.

Let $\mathbb{D}_0$ denote the punctured disc $\mathbb{D}\backslash\{0\}$. Let Σ be a compact Riemann surface of genus g with punctures $p_1, \ldots, p_n$.

Definition 3.1. The class of non-overlapping quasiconformally extendible maps $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ into Σ is the set of n-tuples $(\phi_1, \ldots, \phi_n)$ where

1. For all $i \in \{1, \ldots, n\}$, $\phi_i : \mathbb{D}_0 \to \Sigma$ is holomorphic, and has a quasiconformal extension to a neighborhood of $\overline{\mathbb{D}}$.
2. The continuous extension of ϕ_i takes 0 to p_i
3. For any $i \neq j$, $\overline{\phi_i(\mathbb{D})} \cap \overline{\phi_j(\mathbb{D})}$ is empty.

It was shown in [18] that $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ is a complex Banach manifold.

As in the previous section, we need to refine the class of non-overlapping mappings. We first introduce some terminology. Denote the compactification of a punctured surface Σ by $\overline{\Sigma}$.

Definition 3.2. An n-chart on Σ is a collection of open sets $E_1, \ldots, E_n$ contained in the compactification of Σ such that $E_i \cap E_j$ is empty whenever $i \neq j$, together with local biholomorphic parameters $\zeta_i : E_i \to \mathbb{C}$ such that $\zeta_i(p_i) = 0$.

In the following, we will refer to the charts (ζ_i, E_i) as being on Σ, with the understanding that they are in fact defined on the compactification. Similarly, non-overlapping maps $(f_1, \ldots, f_n)$ will be extended by the removable singularities theorem to the compactification, without further comment.

Definition 3.3. Let $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ be the set of n-tuples of maps $(f_1, \ldots, f_n) \in \mathcal{O}^{\mathrm{qc}}(\Sigma)$ such that for any choice of n-chart $\zeta_i : E_i \to \mathbb{C}$, $i = 1, \ldots, n$ satisfying $\overline{f_i(\mathbb{D})} \subset E_i$ for all $i = 1, \ldots, n$, it holds that $\zeta_i \circ f_i \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

The space $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is well-defined. To see this let (ζ_i, E_i) and (η_i, F_i), $i = 1, \ldots, n$, be n-charts satisfying $\overline{f_i(\mathbb{D})} \subset E_i \cap F_i$ and assume that $\zeta_i \circ f_i \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Since $\eta_i \circ \zeta_i^{-1}$ is holomorphic on an open set containing $\overline{\zeta_i \circ f_i(\mathbb{D})}$, it follows from Lemma 2.5 that $\eta_i \circ f_i = \eta_i \circ \zeta_i^{-1} \circ \zeta_i \circ f_i \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

In order to construct a Hilbert manifold structure on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ we will need some technical theorems.

Theorem 3.4. *Let E be an open neighborhood of 0 in $\mathbb{C}$. Then the set*

$$\left\{ f \in \mathcal{O}^{\mathrm{qc}} : \overline{f(\mathbb{D})} \subset E \right\}$$

is open in $\mathcal{O}^{\mathrm{qc}}$ and the set

$$\left\{ f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{f(\mathbb{D})} \subset E \right\}$$

is open in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

Proof. Let $f_0 \in \mathcal{O}^{\mathrm{qc}}$ satisfy $\overline{f_0(\mathbb{D})} \subset E$. By [18, Corollary 3.5], there exists an open subset W of $\mathcal{O}^{\mathrm{qc}}$ such that $\overline{f(\mathbb{D})} \subset E$ for all $f \in W$. Since f_0 was arbitrary, this proves the first claim.

Now let $f_0 \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ satisfy $\overline{f_0(\mathbb{D})} \subset E$. As above, there exists an open subset W of $\mathcal{O}^{\mathrm{qc}}$ such that $\overline{f(\mathbb{D})} \subset E$ for all $f \in W$. But by Theorem 2.3 $W \cap \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} = \iota^{-1}(W)$ is open in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Thus $\overline{f(\mathbb{D})} \subset E$ for all f in the open set $W \cap \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ containing f_0. This proves the second claim. $\square$

Composition on the left by h is holomorphic operation in both $\mathcal{O}^{\mathrm{qc}}$ and $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. This was proven in [18] in the case of $\mathcal{O}^{\mathrm{qc}}$. The corresponding theorem in the WP-class case is considerably more delicate, and is one of the key theorems necessary to demonstrate the existence of a Hilbert manifold structure on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$. Before we state and prove it we need to investigate some purely analytic issues in the underlying function theory, which will be utilized later.

We start first with the following lemma.

Lemma 3.5. *Let $f_t(z)$ be a holomorphic curve in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ for $t \in N$ where $N \subset \mathbb{C}$ is an open set containing 0. Then there is a domain $N' \subseteq N$ containing 0 and a K which is independent of $t \in N'$ such that*

$$\iint_{\mathbb{D}} |f_t'(z)|^p (1-|z|^2)^\alpha dA \leq K, \tag{3.1}$$

for all $p > 0$ and $\alpha > -1$. The constant K will depend on p and α.

Proof. To establish the estimate (3.1) we observe that since $\mathcal{A}(f_t) \in A^2_1(\mathbb{D})$, $\log f_t'$ is in the little Bloch space; that is

$$\lim_{|z| \to 1^-} (1-|z|^2)|g_t'(z)| = 0,$$

see [23, Corollary 1.4, Chapter 2]. By [11, Theorem 1 (1)], the integral in (3.1) is finite for each t. However, we need a uniform estimate in t. Although this does not follow from the theorem as stated in [11, Theorem 1 (1)], the proof of that theorem can be modified to get the uniform estimate. We proceed by providing the details of this argument. The claim of [11, Theorem 1 (1)] is that

$$g = \log f' \in \mathcal{B}_0 \Longrightarrow \iint_{\mathbb{D}} |f'|^p (1-|z|^2)^\alpha \, dA < \infty \tag{3.2}$$

for all $p > 0$ and $\alpha > -1$ where $\mathcal{B}_0$ is the little Bloch space.

Let $h_s(z) = g(sz)$. This function is continuous on $\overline{\mathbb{D}}$ for $0 < s < 1$. Hence for each fixed s the integral in question converges by an elementary estimate. Therefore (3.2) will follow if we can show that the integral is uniformly bounded for s in some interval $[s_0, 1)$.

We have that $h_s \in \mathcal{B}_0$, that is,

$$\lim_{|z|\to 1-} (1-|z|^2)|h_s'(z)| = 0$$

for all $0 < s \leq 1$. Since $h_1(z) = g(z)$ is in the little Bloch space, and S^1y is compact, given any $\epsilon > 0$ there is an $R > 0$ such that $(1-|z|^2)|h_1'(z)| < \epsilon$ for all $|z| > R$. Fix any $0 < s_0 < 1$ and let $r = R/s_0$. Therefore, if $|z| > r$ and $s_0 < s \leq 1$ then $|sz| > s_0 r = R$ and so for all $|z| > r$ and $s_0 < s \leq 1$ we have $(1-|z|^2)|h_s'(z)| = (1-|z|^2)s|h_1'(sz)| < s\epsilon \leq \epsilon$.

Thus for any $\epsilon > 0$ there are fixed $0 < r < 1$ and $0 < s_0 < 1$ such that

$$(1-|z|^2)|h_s'(z)| < \epsilon \tag{3.3}$$

for all $(s,z) \in [s_0, 1] \times \overline{\mathbb{D}} \backslash D_r$ where $D_r = \{z : |z| < r\}$. Now set

$$\begin{aligned} I &= \iint_{\mathbb{D}} |e^{h_s(z)}|^p (1-|z|^2)^\alpha \, dA, \\ I_1 &= \iint_{D_r} |e^{h_s(z)}|^p (1-|z|^2)^\alpha \, dA, \\ I_2 &= \iint_{\mathbb{D}\backslash D_r} |e^{h_s(z)}|^p (1-|z|^2)^\alpha \, dA. \end{aligned}$$

Our goal is to show that there is a constant C which is independent of $s \in [s_0, 1)$ such that I is bounded by C. It is obvious that this will follow by establishing the aforementioned type of bounds for I_1 and I_2. The estimate for I_1 follows from

$$\begin{aligned} \iint_{D_r} |e^{h_s(z)}|^p (1-|z|^2)^\alpha \, dA &\leq \frac{(1-r^2)^{\min(\alpha,0)}}{s^2} \iint_{D_{rs}} |e^{h_1(z)}|^p \, dA \\ &\leq \frac{(1-r^2)^{\min(\alpha,0)}}{s_0^2} \iint_{D_r} |e^{h_1(z)}|^p \, dA \\ &\leq C. \end{aligned} \tag{3.4}$$

Now we turn to the estimate for I_2. It follows from a theorem of Hardy and Littlewood (see for example [7, Theorem 6] for a proof in the most general case) that there is a C depending only on p and α, such that

$$\iint_{\mathbb{D}} |F(z)|^p (1-|z|^2)^\alpha dA \leq C \left(\iint_{\mathbb{D}} |F'(z)|^p (1-|z|^2)^{p+\alpha} dA + |F(0)|^p \right) \tag{3.5}$$

for $p > 0$ and $\alpha > -1$, whenever at least one of the integrals converges (in fact the two norms represented by each side are equivalent). Now for $s \in [s_0, 1)$ we may apply (3.5) and

(3.3) to $e^{h_s(z)}$ which yield

$$\begin{aligned}
I_2 &\leq \iint_{\mathbb{D}} |e^{h_s(z)}|^p (1-|z|^2)^\alpha \, dA \\
&\leq C \left(\iint_{\mathbb{D}} |e^{h_s(z)}|^p |h_s'(z)|^p (1-|z|^2)^{p+\alpha} dA + |e^{h_s(0)}|^p \right) \\
&\leq C \iint_{\mathbb{D}\setminus D_r} |e^{h_s(z)}|^p |h_s'(z)|^p (1-|z|^2)^{p+\alpha} \, dA + C \iint_{D_r} |e^{h_s(z)}|^p |h_s'(z)|^p (1-|z|^2)^{p+\alpha} \, dA \\
&\quad + C|e^{h_s(0)}|^p \\
&\leq C\epsilon I_2 + C \iint_{D_r} |e^{h_s(z)}|^p |h_s'(z)|^p (1-|z|^2)^{p+\alpha} \, dA + C|e^{h_s(0)}|^p \\
&\leq \frac{1}{2} I_2 + C \iint_{D_r} |e^{h_s(z)}|^p |h_s'(z)|^p (1-|z|^2)^{p+\alpha} \, dA + C|e^{h_s(0)}|^p,
\end{aligned}$$

by choosing $\epsilon \leq \frac{1}{2C}$. Summarizing, we have

$$I_2 \leq 2C \left(\iint_{D_r} |e^{h_s(z)}|^p |h_s'(z)|^p (1-|z|^2)^{p+\alpha} dA + |e^{h_s(0)}|^p \right), \tag{3.6}$$

where r and C are independent of s. Since h_s and h_s' are continuous on $\overline{D_r}$ for $s \in [s_0, 1)$ the integral on the right hand side is bounded by a constant which is independent of $s \in [s_0, 1)$. Therefore the estimates for I_1 and I_2 yield the desired uniform estimate for I. Since the estimate on I is uniform it extends to $s = 1$.

Setting $g_t = \log f_t'$, an argument identical to the above (substituting h_s with g_t) gives the desired uniform bound (3.1) in t, provided that the function $(1-|z|^2)|g_t'(z)|$ is jointly continuous in (t, z). Thus it remains to demonstrate the joint continuity. To this end fix $z_0 \in \overline{\mathbb{D}}$, $t_0 \in N$ and $\epsilon > 0$. There is a δ such that for any $z \in B(z_0, \delta) \cap \overline{\mathbb{D}}$ where $B(z_0, r)$ is the ball of radius δ centered on z_0,

$$\|(1-|z|^2) g_{t_0}'(z) - (1-|z|^2) g_{t_0}'(z_0)\|_\infty < \frac{\epsilon}{2}.$$

Since f_t is a holomorphic curve, there is an interval $(t_0 - \delta_1, t_0 + \delta_1)$ such that

$$\|\mathcal{A}(f_t) - \mathcal{A}(f_{t_0})\| < \epsilon/2.$$

By [23, Lemma 1.3, Chapter II] for $g = \log f'$

$$\|(1-|z|^2) g'(z)\|_\infty \leq \frac{1}{\sqrt{\pi}} \|\mathcal{A}(f)\|$$

(note that in their notation the left hand side is $\|g'(z)\|_\infty$). So for all $z \in \mathbb{D}$ and $t \in (t_0 - \delta_1, t_0 + \delta_1)$,

$$\|(1-|z|^2) g_t'(z) - (1-|z|^2) g_{t_0}'(z)\|_\infty < \frac{\epsilon}{2}. \tag{3.7}$$

Combining this with the fact that $(1-|z|^2) g_t'(z) \to 0$ as $|z| \to 1$ shows that equation (3.7) holds on $\overline{\mathbb{D}}$. Thus, by the triangle inequality

$$\|(1-|z|^2) g_t'(z) - (1-|z|^2) g_{t_0}'(z_0)\|_\infty < \epsilon$$

on $(t_0 - \delta_1, t_0 + \delta_1) \times (D(z_0, r) \cap \overline{\mathbb{D}})$. This proves joint continuity and thus completes the proof. $\square$

Before we state our next lemma we would needs some tools from the theory of Besov spaces which we recall bellow.

Definition 3.6. For $p \in (1,\infty)$, one defines the *Besov space* B^p as the space of holomorphic functions f on $\mathbb{D}$ for which

$$\|f\|_{B^p} = |f(0)| + \left\{ \iint_{\mathbb{D}} |f'(z)|^p \, (1-|z|^2)^{p-2} \, dA \right\}^{\frac{1}{p}} < \infty.$$

From this definition it follows at once that B^2 is the usual Dirichlet space. One also defines for $z \in \mathbb{D}$, the set $S(z)$ by

$$S(z) = \left\{ \zeta \in \mathbb{D} \,:\, 1-|\zeta| \le 1-|z|, \left| \frac{\arg(z\,\overline{\zeta})}{2\pi} \right| \le \frac{1-|z|}{2} \right\}, \tag{3.8}$$

which is obviously a subset of the annulus $|z| \le |\zeta| < 1$.
In our study we shall use the following result, concerning Carleson measures for Besov spaces, due to N. Arcozzi, R. Rochberg and E. Saywer [1].

Theorem 3.7. *Given real numbers p and q with $1 < p < q < \infty$ and a positive Borel measure μ on $\mathbb{D}$, the following two statements are equivalent:*

(1) *There is a constant $C(\mu) > 0$ such that*

$$\|f\|_{L^q(\mu)} \le C(\mu) \|f\|_{B^p}.$$

(2) *For $S(z)$ defined above, one has*

$$\mu(S(z))^{\frac{1}{q}} \le C \left\{ \log \frac{1+|z|}{1-|z|} \right\}^{-\frac{1}{p'}},$$

where p' is the Hölder dual of p.

Using Lemma 3.5 and Theorem 3.7 we can prove the following result:

Lemma 3.8. *Let $f_t(z)$ be a holomorphic curve in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ for $t \in N$ where $N \subset \mathbb{C}$ is an open set containing 0. For any holomorphic function $\psi : \mathbb{D} \to \mathbb{C}$ such that $\iint_{\mathbb{D}} |\psi'|^2 < \infty$ and $\psi(0) = 0$, and any $\beta > 1$, there is a constant C and an open set $N' \subseteq N$ containing 0 such that for all $t \in N'$*

$$\iint_{\mathbb{D}} |f_t'|^2 |\psi|^\beta dA \le C.$$

Proof. The Cauchy-Schwarz inequality and Lemma 3.5 with $p = 4$ and $\alpha = -\frac{1}{2}$ yield

$$\begin{aligned}\iint_{\mathbb{D}} |f_t'(z)|^2 |\psi(z)|^\beta dA &\le \left\{ \iint_{\mathbb{D}} |f_t'(z)|^4 \, (1-|z|^2)^{\frac{-1}{2}} \, dA \right\}^{\frac{1}{2}} \times \left\{ \iint_{\mathbb{D}} |\psi(z)|^{2\beta} \, (1-|z|^2)^{\frac{1}{2}} \, dA \right\}^{\frac{1}{2}} \\ &\le \sqrt{K} \left\{ \iint_{\mathbb{D}} |\psi(z)|^{2\beta} \, (1-|z|^2)^{\frac{1}{2}} \, dA \right\}^{\frac{1}{2}}.\end{aligned}$$

Therefore, since ψ is in the Dirichlet space, to prove that $\iint_{\mathbb{D}} |f_t'(z)|^2 |\psi(z)|^\beta dA \le C$, it would be enough to show that

$$\left\{ \iint_{\mathbb{D}} |\psi(z)|^{2\beta} \, (1-|z|^2)^{\frac{1}{2}} \, dA \right\}^{\frac{1}{2\beta}} \le C' \left\{ \iint_{\mathbb{D}} |\psi'(z)|^2 \, dA \right\}^{\frac{1}{2}}. \tag{3.9}$$

Now, since $\psi(0) = 0$, Theorem 3.7 with $q = 2\beta$, $p = 2$ and $d\mu = (1 - |\zeta|^2)^{\frac{1}{2}}\, dA$, yields that (3.9) holds if and only if for all $z \in \mathbb{D}$

$$\left\{ \iint_{S(z)} (1 - |\zeta|^2)^{\frac{1}{2}}\, dA \right\}^{\frac{1}{2\beta}} \leq C' \left\{ \log \frac{1 + |z|}{1 - |z|} \right\}^{-\frac{1}{2}}. \tag{3.10}$$

Moreover

$$\iint_{S(z)} (1 - |\zeta|^2)^{\frac{1}{2}}\, dA \leq \iint_{|z| \leq |\zeta| < 1} (1 - |\zeta|^2)^{\frac{1}{2}}\, dA = 4\pi \frac{(1 - |z|^2)^{\frac{3}{2}}}{3}.$$

Therefore an elementary calculation yields that (3.10) follows from an estimate of the form

$$(1 - |z|^2)^{\frac{3}{2\beta}} \log \frac{1 + |z|}{1 - |z|} \leq C, \tag{3.11}$$

for all $|z| < 1$. Now if we set $f(r) = (1 - r^2)^{\frac{3}{2\beta}} \log \frac{1+r}{1-r}$ then for all $\varepsilon > 0$, $f(r)$ is continuous on the compact interval $[0, 1 - \varepsilon]$. Indeed the continuity of $f(r)$ is obvious on $[0, 1 - \varepsilon)$ and moreover

$$\lim_{r \to 1^-} (1 - r^2)^{\frac{3}{2\beta}} \log \frac{1 + r}{1 - r} = 0.$$

From this, (3.11) follows and the proof of the lemma is now complete. □

Now we will state and prove the holomorphicity of the operation of left composition in $\mathcal{O}_0^{\mathrm{qc}}$ which will play a crucial role in the establishment of the existence of the Hilbert manifold structure on $\mathcal{O}_0^{\mathrm{qc}}(\Sigma^P)$.

Theorem 3.9. *Let $K \subset \mathbb{C}$ be a compact set which is the closure of an open neighborhood K_{int} of 0 and let A be an open set in $\mathbb{C}$ containing K. If U is the open set*

$$U = \{g \in \mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}} : \overline{g(\mathbb{D})} \subset K_{int}\},$$

and $h : A \to \mathbb{C}$ is a one-to-one holomorphic map such that $h(0) = 0$, then the map $f \mapsto h \circ f$ from U to $\mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}}$ is holomorphic.

Remark 3.10. The fact that U is open follows from Theorem 3.4.

Proof. It was shown in [18, Lemma 3.10] that composition on the left is holomorphic in the above sense on $\mathcal{O}^{\mathrm{qc}}$. However, this does not immediately lead to the desired result, since the norm has changed. Nevertheless some of the computations in [18, Lemma 3.10] can be used here.

As in [18, Lemma 3.10], by Hartogs' theorem [16] it suffices to show that the maps $(\mathcal{A}(f), f'(0)) \mapsto \mathcal{A}(h \circ f)$ and $f'(0) \mapsto h'(0) f'(0)$ are separately holomorphic. The second map is clearly holomorphic. By a theorem in [3, p 198], it suffices to show that $(\mathcal{A}(f), f'(0)) \mapsto \mathcal{A}(h \circ f)$ is Gâteaux holomorphic and locally bounded. It is locally bounded by Lemma 2.5.

To show that this map is Gâteaux holomorphic, consider the curve $(\mathcal{A}(f_0) + t\phi, q(t))$ where $\phi \in A_1^2(\mathbb{D})$ and q is holomorphic in t with $q(0) = f_0'(0)$. It can be easily computed that $(\mathcal{A}(f_t), f_t'(0)) = (\mathcal{A}(f_0) + t\phi, q(t))$ if and only if f_t is the curve

$$f_t(z) = \frac{q(t)}{f_0'(0)} \int_0^z f_0'(u) \exp\left(t \int_0^u \phi(w)dw\right)du.$$

Note that $f_t(z)$ is holomorphic in t for fixed z. Since $\chi(\mathcal{O}^{\mathrm{qc}})$ is open and $\iota : \mathcal{O}_0^{\mathrm{qc}} \to \mathcal{O}^{\mathrm{qc}}$ is continuous, there is an open neighborhood N of 0 in $\mathbb{C}$ such that $f_t \in \mathcal{O}_0^{\mathrm{qc}}$ for all $t \in N$. The neighborhood N can also be chosen small enough that $\overline{f_t(\mathbb{D})} \subset K_{int}$ for all $t \in N$, since we assumed that $t \mapsto f_t$ is a holomorphic curve and the set of $f \in \mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}}$ mapping into K_{int} is open by Theorem 3.4.

Defining $\alpha(t) = \mathcal{A}(h) \circ f_t \cdot f_t'$ and denoting t-differentiation with a dot we then have that

$$\lim_{t\to 0} \frac{1}{t} \left(\mathcal{A}(h \circ f_t) - \mathcal{A}(h \circ f_0)\right) = \dot{\alpha}(t) + \phi.$$

So it is enough to show that

$$\left\| \frac{1}{t} \left(\mathcal{A}(h \circ f_t) - \mathcal{A}(h \circ f_0)\right) - (\dot{\alpha}(t) + \phi) \right\| = \left\| \frac{1}{t} \left(\alpha(t) - \alpha(0) - t\dot{\alpha}(0)\right) \right\| \to 0 \tag{3.12}$$

as $t \to 0$. For any fixed z (recall that $\alpha(t)$ is also a function of z) we have

$$\alpha(t) - \alpha(0) - t\dot{\alpha}(0) = \int_0^t \ddot{\alpha}(s)(t-s)ds.$$

We claim that there is a constant C_0 such that $\|\ddot{\alpha}\| < C_0$ for all t in some neighborhood of 0. Assuming for the moment that this is true, for $|s| < |t| < C$ we set $t = e^{i\theta}u$ and $s = e^{i\theta}v$, and integrating along a ray, we have

$$\begin{aligned}
\|\alpha(t) - \alpha(0) - t\dot{\alpha}(0)\|^2 &= \left\| \int_0^t \ddot{\alpha}(s)(t-s)ds \right\|^2 \\
&= \iint_{\mathbb{D}} \left| \int_0^t \ddot{\alpha}(s)(t-s)\, ds \right|^2 dA \\
&\le \iint_{\mathbb{D}} \left(\int_0^u |\ddot{\alpha}(e^{i\theta}v)|(u-v)dv \right)^2 dA \\
&\le \iint_{\mathbb{D}} \int_0^u u|\ddot{\alpha}(e^{i\theta}v)|^2(u-v)^2 dv\, dA \\
&\le C \iint_{\mathbb{D}} \int_0^u |\ddot{\alpha}(e^{i\theta}v)|^2(u-v)^2 dv\, dA
\end{aligned}$$

where we have used Jensen's inequality and the assumption that $u < C$. Therefore Fubini's theorem and the assumption that $v < u < |t|$ yield

$$\begin{aligned}
\|\alpha(t) - \alpha(0) - t\dot{\alpha}(0)\|^2 &\le 4C|t|^2 \int_0^{|t|} \left(\iint_{\mathbb{D}} |\ddot{\alpha}(s)|^2 dA \right) d|s| \\
&\le C_1 |t|^3.
\end{aligned}$$

Fubini's theorem can be applied since the second to last integral converges by the final inequality. This would prove (3.12). Thus the proof reduces to establishing a bound on $\|\ddot{\alpha}\|$ which is uniform in t in some neighborhood of 0.

By [18, equation 3.2],

$$\begin{aligned} \ddot{\alpha}(t) &= \mathcal{A}(h)'' \circ f_t \cdot f_t' \cdot \dot{f}_t^2 + \mathcal{A}(h)' \circ f_t \cdot f_t' \cdot \ddot{f}_t \\ &\quad +2\mathcal{A}(h)' \circ f_t \cdot \dot{f}_t \cdot \dot{f}_t' + \mathcal{A}(h) \circ f_t \cdot \ddot{f}_t' \\ &= I + II + III + IV \end{aligned} \tag{3.13}$$

where

$$\mathcal{A}(h)' = \frac{h'''}{h'} - \frac{h''^2}{h'^2}$$

and

$$\mathcal{A}(h)'' = \frac{h''''}{h'} - 3\frac{h'''h''}{h'^2} - \frac{h''^3}{h'^3}.$$

We will uniformly bound all the terms on the right side of (3.13) in the $A_1^2(\mathbb{D})$ norm. For all $t \in N$ we have $\overline{f_t(\mathbb{D})} \subset K$ and h is holomorphic on an open set containing the compact set K, and $h' \neq 0$ since h is one-to-one on A. Thus there is a uniform bound for $\mathcal{A}(h)$, $\mathcal{A}(h)'$ and $\mathcal{A}(h)''$ on $f_t(\mathbb{D})$. So by a change of variables, there is an M such that

$$\|\mathcal{A}(h) \circ f_t \cdot f_t'\| = \left(\iint_{f_t(\mathbb{D})} |\mathcal{A}(h)|^2 dA\right)^{1/2} \leq M. \tag{3.14}$$

Similarly there are M' and M'' such that

$$\|\mathcal{A}(h)' \circ f_t \cdot f_t'\| \leq M' \quad \text{and} \quad \|\mathcal{A}(h)'' \circ f_t \cdot f_t'\| \leq M''. \tag{3.15}$$

Since $\overline{f_t(\mathbb{D})}$ is contained in the compact set K, $|f_t(z)|$ is bounded by a constant C which is independent of t. By applying Cauchy estimates in the variable t on a curve $|t| = r_2$, we see that for $0 < r_1 < r_2$ and $|t| \leq r_1$,

$$|\dot{f}_t(z)| \leq \frac{r_2}{(r_1 - r_2)^2} \sup_{|s|=r_2} |f_s(z)|$$

and thus we can find a constant C' such that $|\dot{f}_t(z)| \leq C'$ for $|t| \leq r_1$. Similarly, there is a C'' such that $|\ddot{f}_t(z)| \leq C''$ for all $z \in \mathbb{D}$ and $|t| \leq r_1$. Combining with (3.15), we have that $\|I\|$ and $\|II\|$ are uniformly bounded on $|t| \leq r_1$.

Next, observe that $\|\mathcal{A}(h)' \circ f_t\|_\infty \leq D$ and $\|\mathcal{A}(h) \circ f_t\|_\infty \leq D'$ for some constants D and D' which are independent of t, since $f_t(\mathbb{D})$ is contained inside a compact set in the interior of the domain of h, and h is holomorphic and one-to-one. Therefore, to get a uniform bound on $\|\ddot{\alpha}\|$ we only need to show that $\|\dot{f}_t'\|$ and $\|\ddot{f}_t'\|$ are bounded by some constant which is independent of t on a neighborhood of 0.

A simple computation yields

$$\dot{f}_t'(z) = \frac{\dot{q}(t)}{q(t)} f_t'(z) + \left(\int_0^z \phi(w)dw\right) f_t'(z).$$

Since $q(t)$ is holomorphic and non-zero, $\dot{q}/q$ is uniformly bounded on a neighborhood of 0. Furthermore,

$$\iint_{\mathbb{D}} |f_t'|^2 dA = \text{Area}(f_t(\mathbb{D}))$$

which is uniformly bounded since $f_t(\mathbb{D})$ is contained in a fixed compact set. Since $\psi(z) = \int_0^z \phi(w)dw$ is in the Dirichlet space, we can apply Lemma 3.8 with $\beta = 2$, which proves that $\|\dot{f}_t'\|$ is uniformly bounded for t in some neighborhood of 0. We further compute that

$$\ddot{f}_t'(z) = \frac{\ddot{q}(t)}{q(t)} f_t'(z) + 2\frac{\dot{q}(t)}{q(t)} \left(\int_0^z \phi(w)dw \right) f_t'(z) + \left(\int_0^z \phi(w)dw \right)^2 f_t'(z),$$

so the same reasoning (this time using Lemma 3.8 with $\beta = 2$ and $\beta = 4$) yields a uniform bound for $\|\ddot{f}_t'\|$. This completes the proof. □

4. Complex Hilbert manifold structure on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$

In this section we show that the class of non-overlapping holomorphic maps into a Riemann surface, with WP-class quasiconformal extensions, is a Hilbert manifold. This requires defining a topology and atlas on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$, and proving that this topology is Hausdorff and second countable. Finally we must show that the overlap maps of the atlas are biholomorphisms.

The idea behind the complex Hilbert space structure is as follows. Any element $(f_1, \ldots, f_n)$ of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ maps n closed discs onto closed sets containing the punctures. We choose charts ζ_i, $i = 1, \ldots, n$, which map non-overlapping open neighborhoods of the closed discs into $\mathbb{C}$. The maps $\zeta_i \circ f_i$ are in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, which is an open subset of a Hilbert space. By Theorem 3.4 the components g_i of an element g nearby to f will also have images in the domains of the charts ζ_i. Thus we can model $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ locally by $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Theorem 3.9 will ensure that the transition functions of the charts are biholomorphisms.

We now turn to the proofs, beginning with the topology on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$. Before defining a topological basis we need some notation.

Definition 4.1. For any n-chart $(\zeta, E) = (\zeta_1, E_1, \ldots, \zeta_n, E_n)$ (see Definition 3.2), we say that an n-tuple $U = (U_1, \ldots, U_n) \subset \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, with U_i open in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, is compatible with (ζ, E) if $\overline{f(\mathbb{D})} \subset \zeta_i(E_i)$ for all $f \in U_i$.

For any n-chart (ζ, E) and compatible open subset U of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ let

$$\begin{aligned} V_{\zeta,E,U} &= \{ g \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma) \,:\, \zeta_i \circ g_i \in U_i, \quad i = 1, \ldots, n \} \\ &= \{ (\zeta_1^{-1} \circ h_1, \ldots, \zeta_n^{-1} \circ h_n) \,:\, h_i \in U_i, \quad i = 1, \ldots, n \}. \end{aligned} \tag{4.1}$$

Definition 4.2 (base a for topology on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$)**.** Let

$$\mathcal{V} = \{ V_{\zeta,E,U} \,:\, (\zeta, E) \text{ an } n\text{-chart}, \, U \text{ compatible with } (\zeta, E) \}.$$

Theorem 4.3. *The set $\mathcal{V}$ is the base for a topology on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$. This topology is Hausdorff and second countable.*

Proof. We first establish that $\mathcal{V}$ is a base. For any element f of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$, since $\overline{f_i(\mathbb{D})}$ is compact for all i, there is an n-chart (ζ, E) such that $\overline{f_i(\mathbb{D})} \subset E_i$ for each i. By Theorem 3.4 there is a $U = (U_1, \ldots, U_n)$ compatible with (ζ, E). Thus $\mathcal{V}$ covers $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$.

Now let $V_{\zeta,E,U}$ and $V_{\zeta',E',U'}$ be two elements of $\mathcal{V}$ containing a point $f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$. Define E'' by $E_i'' = E_i \cap E_i'$. For each i choose a compact set κ_i such that $\overline{f_i(\mathbb{D})} \subseteq \kappa_i \subseteq E_i''$. Let $K_i = \zeta_i(\kappa_i)$, $K_i' = \zeta_i'(\kappa_i)$,

$$W_i = \{ \phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{\phi(\mathbb{D})} \subseteq K_i^{int} \}$$

and

$$W_i' = \{ \phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{\phi(\mathbb{D})} \subseteq K_i'^{int} \}$$

where K_i^{int} and $K_i'^{int}$ are the interiors of K_i and K_i' respectively. By Theorem 3.4 W_i and W_i' are open, and by Theorem 3.9 the map $\phi \mapsto \zeta_i' \circ \zeta_i^{-1} \circ \phi$ is a biholomorphism from W_i onto W_i'. So the set

$$U_i'' = U_i \cap \left(\zeta_i \circ \zeta_i'^{-1} \left(W_i' \cap U_i' \right) \right) \subseteq U_i \cap W_i$$

is an open subset of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ (by $\zeta_i'^{-1}(W_i' \cap U_i')$ we mean the set of $\zeta_i'^{-1} \circ \phi$ for $\phi \in W_i' \cap U_i'$). Setting $\zeta_i'' = \zeta|_{E_i''}$ we have that $f \in V_{\zeta'',E'',U''} \subseteq V_{\zeta,E,U} \cap V_{\zeta',E',U'}$ by construction. Thus $\mathcal{V}$ is a base.

To show that the topology generated by $\mathcal{V}$ is Hausdorff, let $f, g \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma^P)$. Choose open, simply connected sets E_i and F_i, $i = 1, \ldots, n$ such that $\overline{f_i(\mathbb{D})} \subset E_i$ and $\overline{g_i(\mathbb{D})} \subset F_i$ and $E_i \cap E_j = F_i \cap F_j = \emptyset$ whenever $i \neq j$. For each i let $\zeta_i : E_i \cup F_i \to \mathbb{C}$ be a biholomorphism taking p_i to 0. Thus $\zeta_i|_{E_i}$ defines an n-chart (ζ, E), and similarly for $\zeta_i|_{F_i}$. (The collection $\zeta_i|_{E_i \cup F_i}$ does not necessarily form an n-chart, but this is inconsequential).

Since $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ is a Hilbert space, it is Hausdorff, so for all i there are open sets U_i and W_i such that $\zeta_i \circ f_i \in U_i$, $\zeta_i \circ g_i \in W_i$, and $U_i \cap W_i = \emptyset$. By Theorem 3.4, by shrinking U_i and W_i if necessary, we can assume that $\overline{h_i(\mathbb{D})} \subset \zeta_i(E_i)$ for all $h_i \in U_i$ and $\overline{h_i(\mathbb{D})} \subset \zeta_i(F_i)$ for all $h_i \in W_i$. That is, U is compatible with (ζ, E) and W is compatible with (ζ, F). Furthermore $f \in V_{\zeta,E,U}$, $g \in V_{\zeta,F,W}$ and $V_{\zeta,E,U} \cap V_{\zeta,F,W} = \emptyset$ by construction. Thus $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is Hausdorff with the topology defined by $\mathcal{V}$.

To see that $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is second countable, we proceed as follows. First observe that Σ is second countable by Rado's Theorem (see for example [13]). Thus it has a countable basis $\mathfrak{B}$ of open sets. Let $\mathfrak{B}^n = \{(B_1, \ldots, B_n)\}$ where each B_i (1) is a finite union of elements of $\mathfrak{B}$ and (2) contains p_i. Clearly $\mathfrak{B}^n$ is countable. Consider the set of n-tuples $C = (C_1, \ldots, C_n)$ such that (1) $(C_1, \ldots, C_n) \in \mathfrak{B}^n$ and (2) $C_i \cap C_j$ is empty whenever $i \neq j$. Since this is a subset of $\mathfrak{B}^n$, it is countable. Furthermore, for each $(C_1, \ldots, C_n)$, we can fix a chart $\zeta_i : C_i \to \mathbb{C}$. Let $\mathfrak{C}$ be the collection of n-charts $\{(\zeta_1, C_1, \ldots, \zeta_n, C_n)\}$ where ζ_i and C_i are as above.

Next, since $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ is a Hilbert space (and hence a separable metric space), it has a countable basis of open sets $\mathfrak{O}$. We define a countable basis for the topology of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ as follows:

$$\mathcal{V}' = \{V_{(\zeta,C,W)} \,:\, (\zeta, C) \in \mathfrak{C},\ W \text{ compatible with } (\zeta, C),\ W_i \in \mathfrak{O},\ i = 1, \ldots, n\}.$$

Each $V' \in \mathcal{V}'$ is open by Theorem 3.4. Furthermore $\mathcal{V}'$ is countable since $\mathfrak{C}$ and $\mathfrak{O}$ are countable. We need to show that $\mathcal{V}'$ is a base for the topology of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$. Clearly $\mathcal{V}' \subset \mathcal{V}$. Thus it is enough to show that for every $f = (f_1, \ldots, f_n) \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ and $V \in \mathcal{V}$ containing f, there is a $V' \in \mathcal{V}'$ such that $f \in V' \subset V$.

Let $V_{\zeta,E,U} \in \mathcal{V}$ contain f. We claim that there is an n-chart $(\eta, C) \in \mathfrak{C}$ such that $\overline{f_i(\mathbb{D})} \subset C_i \subset E_i$ for all i. To see this, fix i and observe that since $\mathfrak{B}$ is a base for Σ, for each point $x \in \overline{f_i(\mathbb{D})}$ there is an open set $B_{i,x} \in \mathfrak{B}$ such that $x \in B_{i,x} \subset E_i$. The set $\{B_{i,x}\}_{x \in \overline{f_i(\mathbb{D})}}$ is a cover of $\overline{f_i(\mathbb{D})}$; since it is compact there is a finite subcover say $\{B_{i,\alpha}\}$. Set $C_i = \cup_\alpha B_{i,\alpha}$ and perform this procedure for each $i = 1, \ldots, n$. By construction the C_i are non-overlapping and $C = (C_1, \ldots, C_n) \in \mathfrak{B}^n$. It follows that $(\eta, C) = (\eta_1, C_1, \ldots, \eta_n, C_n) \in \mathfrak{C}$ where η_i are the charts corresponding to C_i. This proves the claim.

Since $\mathfrak{O}$ is a basis of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, by Theorems 3.4 and 3.9 (using an argument similar to the one earlier in the proof), for each i there is a $W_i \in \mathfrak{O}$ satisfying $\eta_i \circ f_i \in W_i \subset \eta_i \circ \zeta_i^{-1}(U_i)$. If $g \in V'_{\eta,C,W}$ then $g_i = \eta_i^{-1} \circ h_i$ for some $h_i \in W_i$ for all $i = 1, \ldots, n$ by (4.1). But

$h_i \in \eta_i \circ \zeta_i^{-1}(U_i)$, so $g_i \in \zeta_i^{-1}(U_i)$ and hence $g \in V_{\zeta,E,U}$ by (4.1). Thus $V'_{\eta,C,W} \subset V_{\zeta,E,U}$ which completes the proof. $\square$

Remark 4.4. In particular, $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is separable since it is second countable and Hausdorff.

We make one final simple but useful observation regarding the base $\mathcal{V}$.

For a Riemann surface Σ denote by $\mathcal{V}(\Sigma)$ the base for $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ given in Definition 4.2. For a biholomorphism $\rho : \Sigma \to \Sigma_1$ of Riemann surfaces Σ and Σ_1, and for any $V \in \mathcal{V}(\Sigma)$, let

$$\rho(V) = \{\rho \circ \phi \,:\, \phi \in V\}$$

and

$$\rho(\mathcal{V}(\Sigma)) = \{\rho(V) \,:\, V \in \mathcal{V}\}.$$

Theorem 4.5. *If $\rho : \Sigma \to \Sigma_1$ is a biholomorphism between punctured Riemann surfaces Σ and Σ_1 then $\rho(\mathcal{V}(\Sigma)) = \mathcal{V}(\Sigma_1)$.*

Proof. It is an immediate consequence of Definition 4.2 and Theorem 3.9 that $\rho(\mathcal{V}(\Sigma)) \subseteq \mathcal{V}(\Sigma_1)$. Similarly $\rho^{-1}(\mathcal{V}(\Sigma_1)) \subseteq \mathcal{V}(\Sigma)$. Since $\rho(\rho^{-1}(\mathcal{V}(\Sigma_1))) = \mathcal{V}(\Sigma_1)$ and $\rho^{-1}(\rho(\mathcal{V}(\Sigma))) = \mathcal{V}(\Sigma)$ the result follows. $\square$

Definition 4.6 (standard charts on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$)**.** Let (ζ, E) be an n-chart on Σ and let $\kappa_i \subset E_i$ be compact sets containing p_i. Let $K_i = \zeta_i(\kappa_i)$. Let $U_i = \{\psi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \,:\, \overline{\psi(\mathbb{D})} \subset \mathrm{interior}(K_i)\}$. Each U_i is open by Theorem 3.4 and $U = (U_1, \ldots, U_n)$ is compatible with (ζ, E) so we have $V_{\zeta,E,U} \in \mathcal{V}$. A standard chart on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is a map

$$\begin{aligned} T : V_{\zeta,E,U} &\longrightarrow \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \\ (f_1, \ldots, f_n) &\longmapsto (\zeta_1 \circ f_1, \ldots, \zeta_n \circ f_n). \end{aligned}$$

Remark 4.7. To obtain a chart into a Hilbert space, one simply composes with χ as defined by (2.2). Abusing notation somewhat and defining χ^n by

$$\begin{aligned} \chi^n \circ T : V_{\zeta,E,U} &\longrightarrow \bigoplus^n A^2_1(\mathbb{D}) \oplus \mathbb{C} \\ (f_1, \ldots, f_n) &\longmapsto (\chi \circ \zeta_1 \circ f_1, \ldots, \chi \circ \zeta_n \circ g_n) \end{aligned}$$

we obtain a chart into $\bigoplus^n A^2_1(\mathbb{D}) \oplus \mathbb{C}$. Since $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})$ is an open subset of $A^2_1(\mathbb{D}) \oplus \mathbb{C}$ by Theorem 2.3, and χ defines the complex structure $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, we may treat T as a chart with the understanding that the true charts are obtained by composing with χ^n.

Theorem 4.8. *Let Σ be a punctured Riemann surface of type (g, n). With the atlas consisting of the standard charts of Definition 4.6, $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is a complex Hilbert manifold, locally biholomorphic to $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.*

Proof. We have already shown that $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is Hausdorff and separable (in fact second countable). So we need only show that the charts above form an atlas of homeomorphisms with biholomorphic transition functions.

Let $V = V_{\zeta,E,U}$ and $V' = V_{\zeta',E',U'}$ where U and U' are determined by compact sets κ_i and κ'_i respectively, as in Definition 4.6. With the topology from the basis $\mathcal{V}$ of Definition 4.2 the charts are automatically homeomorphisms. It suffices to show that for two standard charts $T : V \to \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ and $T' : V' \to \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ the overlap maps $T \circ T'^{-1}$ and $T' \circ T^{-1}$ are holomorphic.

Assume that $V \cap V'$ is non-empty. For $(\psi_1, \dots, \psi_n) \in T'(V \cap V')$

$$T \circ T'^{-1}(\psi_1, \dots, \psi_n) = (\zeta_1 \circ {\zeta'_1}^{-1} \circ \psi_1, \dots, \zeta_n \circ {\zeta'_n}^{-1} \circ \psi_n).$$

The maps $\psi_i \mapsto \zeta_i \circ {\zeta'_i}^{-1} \circ \psi_i$ are holomorphic maps of $\zeta'_i(V_i \cap V'_i)$ by Theorem 3.9 with $A = \zeta'_i(E_i \cap E'_i)$, $U = \zeta'_i(\zeta_i^{-1}(U_i) \cap {\zeta'_i}^{-1}(U'_i)) = \zeta'_i \circ \zeta_i^{-1}(U_i) \cap U'_i$, $K = \zeta'_i \circ \zeta_i^{-1}(K_i) \cap K'_i$ and $h = \zeta_i \circ {\zeta'_i}^{-1}$. Similarly $T' \circ T^{-1}$ is holomorphic. □

Remark 4.9 (chart simplification). Now that this theorem is proven, we can simplify the definition of the charts. For an n-chart (ζ, E), if we let $U_i = \{f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{f(\mathbb{D})} \subset \zeta_i(E_i)\}$, then the charts T are defined on $V_{\zeta,E,U}$. It is easy to show that T is a biholomorphism on $V_{\zeta,E,U}$, since any $f \in V_{\zeta,E,U}$ is contained in some $V_{\zeta,E,W} \subset V_{\zeta,E,U}$ which satisfies Definition 4.6, and thus T is a biholomorphism on $V_{\zeta,E,W}$ by Theorem 4.8.

Remark 4.10 (standard charts on $\mathcal{O}^{\mathrm{qc}}(\Sigma)$). A standard chart on $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ is defined in the same way as Definition 4.6 and its preamble, by replacing $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ with $\mathcal{O}^{\mathrm{qc}}$ everywhere. Furthermore with this atlas $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ is a complex Banach manifold [18].

Finally, we show that the inclusion map $I : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma) \to \mathcal{O}^{\mathrm{qc}}(\Sigma)$ is holomorphic.

Theorem 4.11. *The complex manifold $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is holomorphically contained in $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ in the sense that the inclusion map $I : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma) \to \mathcal{O}^{\mathrm{qc}}(\Sigma)$ is holomorphic.*

Proof. This follows directly from the construction of the charts on $\mathcal{O}^{\mathrm{qc}}(\Sigma)$. Let $T : V \to \mathcal{O}^{\mathrm{qc}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}$ be a standard chart on $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ as specified in Remark 4.10. Let $U = T(V)$ and $U_0 = U \cap \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Let $V_0 = T^{-1}(U_0)$. The map $T|_{V_0}$ is a chart on $V_0 \subseteq \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$, so it is holomorphic in the WP-class setting. Since the inclusion map $\iota : U_0 \to U$ is holomorphic by Theorem 2.3, the inclusion map $I = T^{-1} \circ \iota \circ \left(T|_{V_0}\right)$ is holomorphic on V_0. Since $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is covered by charts of this form, I is holomorphic. □

References

[1] Arcozzi, N.; Rochberg, R.; and Sawyer, E. *Carleson measures for analytic Besov spaces*, Rev. Mat. Iberoamericana **18** (2002), no. 2, 443–510.

[2] Astala, K. and Zinsmeister, M. *Teichmüller spaces and BMOA*, Mathematische Annalen, **289** no 1 (1991), 613–625.

[3] Chae, S. B. *Holomorphy and Calculus in Normed Spaces.* With an appendix by Angus E. Taylor. Monographs and Textbooks in Pure and Applied Mathematics, 92. Marcel Dekker, Inc., New York, 1985.

[4] G. Cui, Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces, Sci. China Ser. A 43 (2000), no. 3, 267–279.

[5] Cui, G. and Zinsmeister, M. *BMO-Teichmüller spaces.* Illinois Journal of Mathematics **48** no. 4 (2004), 1223–1233.

[6] Fritzsche, K. and Grauert, H. *From Holomorphic Functions to Complex Manifolds*, Graduate Texts in Mathematics, **213**, Springer-Verlag, New York, 2002.

[7] Flett, T. M. *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. **38** (1972), 746–765.

[8] Friedan, D. and Shenker, S. *The analytic geometry of two-dimensional conformal field theory.* Nuclear Physics B **281** (1987), 509–545.

[9] Gardiner, F. P. *Schiffer's interior variation and quasiconformal mapping*, Duke Math. J. **42** (1975), 371–380.

[10] Gardiner, F. P. and Sullivan, D. P. *Symmetric structures on a closed curve.* American Journal of Mathematics, **114** no. 4 (1992), 683–736.

[11] Gallardo-Gutiérrez, E., González, M.J., Pérez-González, F., Pommerenke, F. and Rättyä, J. *Locally univalent functions, VMOA and the Dirichlet space*, Centre de Recerca Matemàtica preprint crm-pr898.

[12] Grosse-Erdmann, K.-G. *A weak criterion for vector-valued holomorphy.* Math. Proc. Cambridge Philos. Soc., **169** no. 2 (2004), 399–411.

[13] Hubbard, J. H., with contributions by Douady, A., Dunbar, W., Roeder, R., Bonnot, S., Brown, D. Hatcher, A. Hruska, C. and Mitra, S. Teichmüller theory and applications to geometry, topology and dynamics. Volume 1: Teichmüller theory. 2006 Matrix Editions, Ithaca, NY.

[14] Hui, G. *Integrable Teichmüller spaces.* Science in China, **43** no 1, Series A (2000), 47–58.

[15] Lehto, O. and Virtanen, K. I. *Quasiconformal Mappings in the Plane.* 2nd edition. Die Grundlehren der mathematischen Wissenshcaften, Band 126. Springer-Verlag, New York-Heidelberg, 1973.

[16] Mujica, J. *Complex Analysis in Banach Spaces.* North Holland, 1986.

[17] Radnell, D. and Schippers, E. *Quasisymmetric sewing in rigged Teichmueller space*, Commun. Contemp. Math. **8** no 4 (2006), 481–534.

[18] Radnell, D. and Schippers, E. *A complex structure on the set of quasiconformally extendible non-overlapping mappings into a Riemann surface*, J. Anal. Math. **108** (2009), 277–291

[19] Radnell, D. and Schippers, E., *Fiber structure and local coordinates for the Teichmüller space of a bordered Riemann surface*, Conform. Geom. Dyn. **14** (2010), 14–34.

[20] Radnell, D., Schippers, E.; and Staubach, W. *Dirichlet space of multiply connected domains with Weil-Petersson class boundaries.* Preprint 2013.

[21] Segal, G. *The definition of conformal field theory*, Topology, Geometry and Quantum Field Theory (U. Tillmann, ed.), London Mathematical Society Lecture Note Series, vol. 308, Cambridge University Press, 2004, Original preprint 1988, pp. 421–576.

[22] Y. Shen, Weil-Petersson Teichmüller space, prepint, arXiv:1304.3197.

[23] Tatkhajan, L. A. and Teo, L.-P. *Weil-Petersson Metric on the Universal Teichmüller Space.* Memoirs of the American Mathematical Society. **183** no 861 (2006).

[24] Teo, L.-P. *Velling-Kirillov metric on the universal Teichmüller curve.* J. Anal. Math. **93** (2003), 271–307.

[25] Vafa, C. *Conformal theories and punctured surfaces*, Phys. Lett. B, **199** no. 2, (1987), 195–202.

David Radnell, Department of Mathematics and Statistics, American University of Sharjah, PO BOX 26666, Sharjah, United Arab Emirates

E-mail address: dradnell@aus.edu

Eric Schippers, Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, R3T 2N2, Canada

E-mail address: eric_schippers@umanitoba.ca

Wolfgang Staubach, Department of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden

E-mail address: wulf@math.uu.se