A HILBERT MANIFOLD STRUCTURE ON THE WEIL-PETERSSON CLASS TEICHMÜLLER SPACE OF BORDERED RIEMANN SURFACES

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Abstract. We consider bordered Riemann surfaces which are biholomorphic to compact Riemann surfaces of genus g with n regions biholomorphic to the disk removed. We define a refined Teichmüller space of such Riemann surfaces (which we refer to as the WP-class Teichmüller space) and demonstrate that in the case that $2g + 2 - n > 0$, this refined Teichmüller space is a Hilbert manifold. The inclusion map from the refined Teichmüller space into the usual Teichmüller space (which is a Banach manifold) is holomorphic.

We also show that the rigged moduli space of Riemann surfaces with non-overlapping holomorphic maps, appearing in conformal field theory, is a complex Hilbert manifold. This result requires an analytic reformulation of the moduli space, by enlarging the set of non-overlapping mappings to a class of maps intermediate between analytically extendible maps and quasiconformally extendible maps. Finally we show that the rigged moduli space is the quotient of the refined Teichmüller space by a properly discontinuous group of biholomorphisms.

1. Introduction

In this paper, we construct a refinement of the Teichmüller space of bordered Riemann surfaces of genus g with n boundary curves homeomorphic to the circle, which we will refer to as the Weil-Petersson class Teichmüller space. If $2g + 2 - n > 0$ this Weil-Petersson class Teichmüller space possesses a Hilbert manifold structure, and furthermore the inclusion map from this Teichmüller space into the standard one is holomorphic. Using the results of the present paper, the authors showed that the Teichmüller space in this paper possesses a convergent Weil-Petersson metric [26]. (This justifies the term "Weil-Petersson class", which we will often abbreviate as "WP-class").

Our approach can be summarized as follows: we combine the results of L. Takhtajan and L.-P. Teo [29] and G. Hui [11] refining the universal Teichmüller space, with the results of D. Radnell and E. Schippers [21, 22, 23] demonstrating the relation between a moduli space in conformal field theory and the Teichmüller space of bordered surfaces. We also require a result by S. Nag [17, 18] on the variational method of F. Gardiner and M. Schiffer [9], together with the theory of marked holomorphic families of Riemann surfaces (see for example [6, 14, 18]). An essential part of our approach is the utilization of a kind of fibration of the (infinite-dimensional) Teichmüller space of bordered surfaces over the (finite-dimensional) Teichmüller space of compact Riemann surfaces with punctures discovered by the first two authors in [23]. Each fiber is a collection of n-tuples of non-overlapping maps into a fixed punctured Riemann surfaces, of a certain regularity. The authors demonstrated that, in the Weil-Petersson class case, the fibers are complex Hilbert manifolds [24, 25].

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Our investigations are motivated both by Teichmüller theory (see below), and by conformal field theory, where our results are required to solve certain analytic problems in the construction of conformal field theory from vertex operator algebras following Y.-Z. Huang [12]. First, we give some background for the problem, and then outline our approach.

There has been interest in refinements of quasiconformal Teichmüller space for some time [2, 5, 10]. It was previously observed by S. Nag and A. Verjovsky [19] that the Weil-Petersson metric diverges on the Bers universal Teichmüller space except on a subspace of the tangent space. A family of L^p-class universal Teichmüller spaces was given by Hui [11], who attributed the L^2 case to G. Cui [4]. The L^2 case has now come to be called the "Weil-Petersson class universal Teichmüller space" [28], since this universal Teichmüller space possesses a convergent Weil-Petersson metric. Independently, Takhtajan and Teo [29] defined a Hilbert manifold structure on the universal Teichmüller space and the universal Teichmüller curve, equivalent to that of Hui, and furthermore showed that it is a topological group. They also obtained potentials for the Weil-Petersson metric and investigated its relation to the Kirillov-Yuri'ev-Nag-Sullivan period map, a holomorphic embedding of the universal Teichmüller space via the period map, and its relation to the generalized Grunsky matrix, among other results. Using the results of the present paper, the authors demonstrated that the Teichmüller space of bordered surfaces studied in this paper possesses a convergent Weil-Petersson metric [26], thus generalizing some of the results of [4, 11, 29].

The other motivation comes from conformal field theory, where one considers a moduli space of Riemann surfaces with extra data, originating with D. Friedan and S. Shenker [8]. We will use two different formulations of this moduli space due to G. Segal [27] and C. Vafa [31]. Vafa's *puncture model* of the rigged moduli space consists of equivalence classes of pairs (Σ, ϕ), where Σ is a compact Riemann surface with n punctures, and $\phi = (\phi_1, \dots, \phi_n)$ is an n-tuple of one-to-one holomorphic maps from the unit disk $\mathbb{D} \subset \mathbb{C}$ into the Riemann surface with non-overlapping images. Two such pairs (Σ_1, ϕ) and (Σ_2, ψ) are equivalent if there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $\psi_i = \sigma \circ \phi_i$ for $i = 1, \dots, n$. The n-tuple of maps $(\phi_1, \dots, \phi_n)$ is called the rigging, and is usually subject to some additional regularity conditions which vary in the conformal field theory literature. The choice of these regularity conditions relates directly to the analytic structure of this moduli space. The regularity also relates directly to the regularity of certain elliptic operators, which are necessary for the rigorous definition of conformal field theory in the sense of Segal [27]. In this paper we show that the rigged moduli space has a Hilbert manifold structure, and that this Hilbert manifold structure arises naturally from a refined Teichmüller space of bordered surfaces, which we also show is a Hilbert manifold. These results are further motivated by the fact that the aforementioned elliptic operators will have convergent determinants on precisely this refined moduli space. We hope to return to this question in a future publication. Moreover, these results will have applications to the construction of higher genus conformal field theory, following a program of Huang and others [12, 13].

These results are made possible by previous work of two of the authors [22], in which it was shown that if one chooses the riggings to be extendible to quasiconformal maps of a neighborhood of the closure of $\mathbb{D}$, then the rigged moduli space is the same as the Teichmüller space of a bordered Riemann surface (up to a properly discontinuous group action). Thus the rigged moduli space inherits a complex Banach manifold structure from Teichmüller space. This solved certain analytic problems in the definition of conformal field theory, including holomorphicity of the sewing operation.

On the other hand this also provided an alternate description of the Teichmüller space of a bordered surface Σ as a fiber space that is locally modeled on the following rigged Teichmüller space. In [23] (following the first author's thesis [20]), two of the authors introduced the *rigged Teichmüller space* based on quasiconformally extendible riggings, which is the analogue of the above rigged moduli space. It was proved that this rigged Teichmüller space is a fiber space: the fibers consist of non-overlapping maps into a compact Riemann surface with punctures obtained by sewing copies of the punctured disk onto the boundaries of Σ. The base space is the finite-dimensional Teichmüller space of the compact surface with punctures so obtained.

Thus the Teichmüller space of bordered surfaces has two independent complex Banach manifolds structures: the standard one, obtained from the Bers embedding of spaces of equivalent Beltrami differentials, and one obtained from the fiber model. It was shown that the two are equivalent [22, 23]. The fibers are a natural function space of quasiconformally extendible conformal maps with non-overlapping images (these are also the riggings described above). In [24, 25] we use the results of Hui [11] and Takhtajan and Teo [29] to show that if one restricts to WP-class non-overlapping mappings then the collection of riggings is a Hilbert manifold. Here we define the WP-class rigged Teichmüller space and prove that it is a Hilbert manifold by using the fiber structure and the aforementioned results. Finally, we define a refined Teichmüller space of bordered surfaces and, via the fiber model, show that it is a Hilbert manifold using the refined rigged Teichmüller space. Charts for the refined Teichmüller space will be defined completely explicitly, using Gardiner-Schiffer variation and natural function spaces of non-overlapping maps.

The proof that these charts define a Hilbert manifold structure is somewhat complicated. We proceed in the following way. In Section 2, we define the refined quasiconformal mappings and function spaces which will appear in the paper. This section mostly establishes notation and outlines some previous results, and proves some elementary facts about the refined mappings. In Section 2.4, we define the set of WP-class non-overlapping mappings which serves as a model of the fibers, and recall the construction of the holomorphic atlas from [24, 25]. In Section 3, we show that the WP-class rigged Teichmüller space is a Hilbert manifold. We do this using the results of the previous section, and Gardiner-Schiffer variation. A key part of the argument relies on the universality properties of the universal Teichmüller curve and the theory of marked holomorphic families of Riemann surfaces. Finally, in Section 4 we show that the WP-class Teichmüller space of a bordered Riemann surface is a Hilbert manifold, by showing that it covers the refined rigged Teichmüller space and passing the structure upwards. Furthermore, we show that the Hilbert manifold structure passes downwards to the two versions of the rigged moduli space of conformal field theory defined by Segal [27] and Vafa [31].

2. Definitions and results on WP-class mappings

In Section 2.1 we collect some known results on the refinement of the set of quasisymmetries and quasiconformal maps, from the work of Takhtajan and Teo [29], Teo [30] and Hui [11]. We also collect some theorems of the authors which will be necessary in the rest of the paper [24, 25]. In Section 2.2 we define the WP-class quasisymmetries between borders of Riemann surfaces in an obvious way and some elementary results are derived. This is then used to define the WP-class quasiconformal maps between Riemann surfaces in Section 2.3.

Finally, in Section 2.4 we recall the definition of the class of non-overlapping WP-class maps and the main theorems regarding the Hilbert manifold structure on them, obtained in [24, 25]. These non-overlapping maps are fibered over the finite-dimensional Teichmüller space of punctured Riemann surfaces. We will use this fact to construct the Hilbert manifold structure of the WP-class Teichmüller space.

2.1. **WP-class maps on the disk and circle.** In this section we give the definitions of Weil-Petersson class (henceforth WP-class) conformal maps of the disk and quasisymmetries of the circle. We also state some of the fundamental results regarding these, given in the theory of the Weil-Petersson universal Teichmüller space of Takhtajan and Teo [29] and Guo Hui [11]. We also state some results obtained by the authors in [24, 25] which will be essential to the main results of this paper.

In [22] we defined the set $\mathcal{O}^{\mathrm{qc}}$ of quasiconformally extendible maps in the following way.

Definition 2.1. Let $\mathcal{O}^{\mathrm{qc}}$ be the set of maps $f : \mathbb{D} \to \mathbb{C}$ such that f is one-to-one, holomorphic, has quasiconformal extension to $\mathbb{C}$, and $f(0) = 0$.

A Banach space structure can be introduced on $\mathcal{O}^{\mathrm{qc}}$ as follows. Let

$$A_1^\infty(\mathbb{D}) = \left\{ \phi \in \mathcal{H}(\mathbb{D}) : \|\phi\|_{A_1^\infty(\mathbb{D})} = \sup_{z\in\mathbb{D}} (1-|z|^2)|\phi(z)| < \infty \right\}. \tag{2.1}$$

This is a Banach space. It follows directly from results of Teo [30] that for

$$\mathcal{A}(f) = \frac{f''}{f'}$$

the map

$$\begin{aligned} \chi : \mathcal{O}^{\mathrm{qc}} &\longrightarrow A_1^\infty(\mathbb{D}) \oplus \mathbb{C} \\ f &\longmapsto (\mathcal{A}(f), f'(0)) \end{aligned} \tag{2.2}$$

takes $\mathcal{O}^{\mathrm{qc}}$ onto an open subset of the Banach space $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$ (see [22]). Thus $\mathcal{O}^{\mathrm{qc}}$ inherits a complex structure from $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$.

The space $\mathcal{O}^{\mathrm{qc}}$ can be thought of as a two-complex-dimensional extension of the universal Teichmüller space. We will construct a Hilbert structure on a subset of $\mathcal{O}^{\mathrm{qc}}$. To do this, in place of $A_1^\infty(\mathbb{D})$ we use the Bergman space

$$A_1^2(\mathbb{D}) = \left\{ \phi \in \mathcal{H}(\mathbb{D}) : \|\phi\|_2^2 = \iint_{\mathbb{D}} |\phi|^2 \, dA < \infty \right\}$$

which is a Hilbert space and a vector subspace of the Banach space $A_1^\infty(\mathbb{D})$. Furthermore, the inclusion map from $A_1^2(\mathbb{D})$ to $A_1^\infty(\mathbb{D})$ is bounded [29, Chapter II Lemma 1.3]. Here and in the rest of the paper we shall denote the Bergman space norm $\|\cdot\|_{A_1^2}$ by $\|\cdot\|$.

We define the class of WP quasiconformally extendible maps as follows.

Definition 2.2. Let

$$\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} = \left\{ f \in \mathcal{O}^{\mathrm{qc}} : \mathcal{A}(f) \in A_1^2(\mathbb{D}) \right\}.$$

We will embed $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ in the Hilbert space direct sum $\mathcal{W} = A_1^2(\mathbb{D}) \oplus \mathbb{C}$. Since $\chi(\mathcal{O}^{\mathrm{qc}})$ is open, $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}) = \chi(\mathcal{O}^{\mathrm{qc}}) \cap A_1^2(\mathbb{D})$ is also open, and thus $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ trivially inherits a Hilbert manifold structure from $\mathcal{W}$. We summarize this with the following theorem.

Theorem 2.3 ([24, 25])**.** *The inclusion map $A_1^2(\mathbb{D}) \to A_1^\infty(\mathbb{D})$ is continuous. Furthermore $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})$ is an open subset of the vector subspace $\mathcal{W} = A_1^2(\mathbb{D}) \oplus \mathbb{C}$ of $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$, and the inclusion map from $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})$ to $\chi(\mathcal{O}^{\mathrm{qc}})$ is holomorphic. Thus the inclusion map $\iota : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \to \mathcal{O}^{\mathrm{qc}}$ is holomorphic.*

Lemma 2.4 ([24, 25])**.** *Let $f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Let h be a one-to-one holomorphic map defined on an open set W containing $\overline{f(\mathbb{D})}$. Then $h \circ f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Furthermore, there is an open neighborhood U of f in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ and a constant C such that $\|\mathcal{A}(h \circ g)\| \leq C$ for all $g \in U$.*

We will also need a technical lemma on a certain kind of holomorphicity of left composition in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

Lemma 2.5 ([24, 25])**.** *Let E be an open subset of $\mathbb{C}$ containing 0 and Δ an open subset of $\mathbb{C}$. Let $\mathcal{H} : \Delta \times E \to \mathbb{C}$ be a map which is holomorphic in both variables and injective in the second variable. Let $h_\epsilon(z) = \mathcal{H}(\epsilon, z)$ and let $\psi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ satisfy $\overline{\psi(\mathbb{D})} \subseteq E$. Then the map $Q : \Delta \mapsto \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ defined by $Q(\epsilon) = h_\epsilon \circ \psi$ is holomorphic in ϵ.*

Next, we define a subset $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ of the quasisymmetries in the following way. Briefly, a map $h : \mathbb{S}^1 \to \mathbb{S}^1$ is in $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ if the corresponding welding maps are in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Let $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$, and let $\mathrm{QS}(\mathbb{S}^1)$ be the set of quasisymmetric maps from S^1 to S^1. For $h \in \mathrm{QS}(\mathbb{S}^1)$ let $w_\mu(h) : \mathbb{D}^* \to \mathbb{D}^*$ be a quasiconformal extension of h with dilatation μ (such an extension exists by the Ahlfors-Beurling extension theorem). Furthermore, let $w^\mu : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be the quasiconformal map with dilatation μ on $\mathbb{D}^*$ and 0 on $\mathbb{D}$, with normalization $w^\mu(0) = 0$, $w^{\mu\prime}(0) = 1$ and $w^\mu(\infty) = \infty$ and set

$$F(h) = w^\mu|_{\mathbb{D}}\,.$$

It is a standard fact that $F(h)$ is independent of the choice of extension w_μ.

Definition 2.6. We define a subset of $\mathrm{QS}(\mathbb{S}^1)$ by

$$\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1) = \{h \in \mathrm{QS}(\mathbb{S}^1) \,:\, F(h) \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}\}.$$

Remark 2.7. A change in the normalization of $w^{\mu\prime}(0)$ results in exactly the same set.

An alternate characterization of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ follows from a theorem proved by Guo Hui [11]. Let

$$L^2_{\mathrm{hyp}}(\mathbb{D}^*) = \left\{ \mu : \iint_{\mathbb{D}^*} (|z|^2 - 1)^{-2} |\mu(z)|^2 dA < \infty \right\},$$

and let

$$L^\infty(\mathbb{D}^*)_1 = \{\mu : \mathbb{D}^* \to \mathbb{C} : \|\mu\|_\infty \leq k \text{ for some } k < 1\}$$

(that is, the unit ball in $L^\infty(\mathbb{D}^*)$). Note that the line element of the hyperbolic metric on $\mathbb{D}$ is $|dz|(1 - |z|^2)^{-1}$ and the line element of the hyperbolic metric on $\mathbb{D}^*$ is $|dz|(|z|^2 - 1)^{-1}$. Thus the above condition says that μ is L^2 with respect to hyperbolic area. The following two theorems follow from Theorems 1 and 2 of [11].

Theorem 2.8 (Hui)**.** *Let f be a one-to-one holomorphic function on $\mathbb{D}$ such that $f(0) = 0$. Then $f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ if and only if there exists a quasiconformal extension $\tilde{f}$ of f to $\mathbb{C}$ whose dilatation μ is in $L^2_{\mathrm{hyp}}(\mathbb{D}^*) \cap L^\infty(\mathbb{D}^*)_1$.*

Theorem 2.9 (Hui)**.** *Let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be a quasisymmetry. Then $\phi \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ if and only if there is a quasiconformal extension $h : \mathbb{D}^* \to \mathbb{D}^*$ of ϕ such that the Beltrami differential $\mu(h)$ of h is in $L^2_{\mathrm{hyp}}(\mathbb{D}^*)$.*

It follows from Theorem 1.12 of Part II and Lemma 3.4 of Part I of [29] that $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ is a group.

Theorem 2.10 (Takhtajan-Teo). *The set* $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ *is closed under composition and inversion.*

By an analytic map $h : \mathbb{S}^1 \to \mathbb{S}^1$ we mean that h is the restriction of an analytic map of a neighborhood of $\mathbb{S}^1$. Let $\mathbb{A}(r, s)$ denote the annulus $\{z : r < |z| < s\}$ and $D(z_0, r)$ denote the disk $\{z : |z - z_0| < r\}$.

Proposition 2.11. *If* $h : \mathbb{S}^1 \to \mathbb{S}^1$ *is one-to-one and analytic, then* h *has a quasiconformal extension to* $\mathbb{D}^*$ *which is holomorphic in an annulus* $\mathbb{A}(1, R)$ *for some* $R > 1$*. Furthermore* $h \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$*.*

Proof. To prove the first claim, observe that h has an analytic extension $\tilde{h}$ to some annulus $\mathbb{A}(r, s)$ for $r < 1 < s$. Let R be such that $1 < R < s$. Applying the Ahlfors-Beurling extension theorem to the circle $|z| = R$, there exists a quasiconformal map $g : \mathbb{A}(R, \infty) \to \mathbb{A}(R, \infty)$ whose boundary values agree with $\tilde{h}$ restricted to $|z| = R$. Let H be the map which is equal to $\tilde{h}$ on $\mathbb{A}(1, R)$ and g on $\mathbb{A}(R, \infty)$. Then H is quasiconformal on $\mathbb{D}^*$ since it is quasiconformal on the two pieces and continuous on $\mathbb{D}$ (see [15, V.3]). Thus, H has the desired properties.

The second claim follows from Theorem 2.8 since the dilatation of H is zero in $\mathbb{A}(1, R)$. □

2.2. **Refined quasisymmetric mappings between boundaries of Riemann surfaces.** We first clarify the meaning of "bordered Riemann surface". By a half-disk, we mean a set of the form $\{z : |z - z_0| < r \text{ and } \mathrm{Im}(z) \geq 0\}$ for some z_0 on the real axis. By a bordered Riemann surface, we mean a Riemann surface with boundary, such that for every point on the boundary there is a homeomorphism of a neighborhood of that point onto a half-disk. It is further assumed that for any pair of charts ρ_1, ρ_2 whose domains overlap, the map $\rho_2 \circ \rho_1^{-1}$ and its inverse is a one-to-one holomorphic map on its domain. Note that this implies, by the Schwarz reflection principle, that $\rho_2 \circ \rho_1^{-1}$ extends to a one-to-one holomorphic map of an open set containing the portion of the real axis in the domain of the original map. Every bordered Riemann surface has a double which is defined in the standard way. See for example [1].

Following standard terminology (see for example [18]) we say that a Riemann surface is of *finite topological type* if its fundamental group is finitely generated. A Riemann surface is said to be of finite topological type (g, n, m) if it is biholomorphic to a compact genus g Riemann surface with n points and m parametric disks removed. By a parametric disk we mean a region biholomorphic to the unit disk.

In this paper we will be entirely concerned with Riemann surfaces of type $(g, 0, n)$ and $(g, n, 0)$ and we will use the following terminology. A *bordered Riemann surface* of type (g, n) will refer to a bordered Riemann surface of type $(g, 0, n)$ and a *punctured Riemann surface* of type (g, n) will refer to a Riemann surface of type $(g, n, 0)$. It is furthermore assumed that the boundary curves and punctures are given a numerical ordering. Finally, a *boundary curve* will be understood to mean a connected component of the boundary of a bordered Riemann surface. Note that each boundary curve is homeomorphic to $\mathbb{S}^1$.

Remark 2.12. Any quasiconformal map between bordered Riemann surfaces has a unique continuous extension taking the boundary curves to the boundary curves. To see this let Σ_1^B and Σ_2^B be bordered Riemann surfaces, and let Σ_1^d and Σ_2^d denote their doubles. By reflecting, the quasiconformal map extends to the double: the reflected map is continuous on Σ_1^d, takes Σ_1^d onto Σ_2^d, and is quasiconformal on the double minus the boundary curves. Since each boundary curve of Σ_i^B is an analytic curve in the double, the map is quasiconformal on Σ_1^d [15, V.3] and in particular continuous on each analytic curve.

Throughout the paper, we will label the original map and its continuous extension with the same letter to avoid complicating the notation. When referring to a "bordered Riemann surface", we will be referring to the interior. However, in the following all maps between bordered Riemann surfaces will be at worst quasiconformal and thus by Remark 2.12 have unique continuous extensions to the boundary. Thus the reader could treat the border as included in the Riemann surface with only trivial changes to the statements in the rest of the paper.

Definition 2.13. Let Σ^B be a bordered Riemann surface and C be one of its boundary components. A *collar neighborhood of* C is an open set U which is biholomorphic to an annulus, and one of whose boundary curves is C. A *collar chart of* C is a biholomorphism $H : U \to \mathbb{A}(1, r)$ where U is a collar neighborhood of C, whose continuous extension to C maps C to $\mathbb{S}^1$.

Note that any collar chart must have a continuous one-to-one extension to C, which maps C to $\mathbb{S}^1$. (In fact application of the Schwarz reflection principle shows that H must have a one-to-one holomorphic extension to an open tubular neighborhood of C in the double of Σ.) We may now define the class of WP quasisymmetries between boundary curves of bordered Riemann surfaces.

Definition 2.14. Let Σ_1^B and Σ_2^B be bordered Riemann surfaces, and let C_1 and C_2 be boundary curves of Σ_1^B and Σ_2^B respectively. Let $\mathrm{QS}_{\mathrm{WP}}(C_1, C_2)$ denote the set of orientation-preserving homeomorphisms $\phi : C_1 \to C_2$ such that there are collar charts H_i of C_i, $i = 1, 2$ respectively, such that $H_2 \circ \phi \circ H_1^{-1}\big|_{\mathbb{S}^1} \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$.

Remark 2.15. The notation $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, C_1)$ will always be understood to refer to $\mathbb{S}^1$ as the boundary of an annulus $\mathbb{A}(1, r)$ for $r > 1$. We will also write $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1) = \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \mathbb{S}^1)$.

Proposition 2.16. *If $\phi \in \mathrm{QS}_{\mathrm{WP}}(C_1, C_2)$ then for any pair of collar charts H_i of C_i, $i = 1, 2$ respectively,* $H_2 \circ \phi \circ H_1^{-1}\big|_{\mathbb{S}^1} \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$.

Proof. Assume that there are collar charts H_i' of C_i such that $H_2' \circ \phi \circ {H_1'}^{-1} \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$. Let H_i be any other pair of collar charts. The composition

$$H_2 \circ {H_2'}^{-1} \circ H_2' \circ \phi \circ {H_1'}^{-1} \circ H_1' \circ H_1^{-1} = H_2 \circ \phi \circ H_1^{-1}$$

is defined on some collar neighborhood of C_1. Since $H_2 \circ {H_2'}^{-1}$ and $H_1' \circ H_1^{-1}$ have analytic extensions to $\mathbb{S}^1$, the result follows from Proposition 2.11 and Theorem 2.10. □

Proposition 2.17. *Let Σ_i^B be bordered Riemann surfaces and C_i a boundary curve on each surface for $i = 1, 2, 3$. If $\phi \in \mathrm{QS}_{\mathrm{WP}}(C_1, C_2)$ and $\psi \in \mathrm{QS}_{\mathrm{WP}}(C_2, C_3)$ then $\psi \circ \phi \in \mathrm{QS}_{\mathrm{WP}}(C_1, C_3)$.*

Proof. Let H_i be collar charts of C_i for $i = 1, 2, 3$. In that case

$$H_3 \circ \psi \circ \phi \circ H_1^{-1} = H_3 \circ \psi \circ H_2^{-1} \circ H_2 \circ \phi \circ H_1^{-1}$$

when restricted to C_1. By Proposition 2.16 both $H_3 \circ \psi \circ H_2^{-1}$ and $H_2 \circ \phi \circ H_1^{-1}$ are in $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$, so the composition is in $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ by Theorem 2.10. Thus $\psi \circ \phi \in \mathrm{QS}_{\mathrm{WP}}(C_1, C_3)$ by definition. □

2.3. **A WP-class of quasiconformal mappings between bordered surfaces.** We can now define a WP-class of quasiconformal mappings.

Definition 2.18. Let Σ_1^B and Σ_2^B be bordered Riemann surfaces of type (g,n), with boundary curves C_1^i and C_2^j $i=1,\ldots,n$ and $j=1,\ldots,n$ respectively. The class of maps $\mathrm{QC}_0(\Sigma_1^B,\Sigma_2^B)$ consists of those quasiconformal maps from Σ_1^B onto Σ_2^B such that the continuous extension to each boundary curve C_1^i, $i=1,\ldots,n$ is in $\mathrm{QS}_{\mathrm{WP}}(C_1^i,C_2^j)$ for some $j\in\{1,\ldots,n\}$.

Note that the continuous extension to a boundary curve C_1^i *must* map onto a boundary curve C_2^j.

The following two Propositions follow immediately from Definition 2.18 and Proposition 2.17.

Proposition 2.19. *Let Σ_i^B $i=1,2,3$ be bordered Riemann surfaces of type (g,n). If $f\in \mathrm{QC}_0(\Sigma_1^B,\Sigma_2^B)$ and $g\in\mathrm{QC}_0(\Sigma_2^B,\Sigma_3^B)$ then $g\circ f\in\mathrm{QC}_0(\Sigma_1^B,\Sigma_3^B)$.*

Proposition 2.20. *Let Σ_1^B and Σ_2^B be bordered Riemann surfaces. Let C_1 be a boundary curve of Σ_1^B, $\phi\in\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1,C_1)$, $f\in\mathrm{QC}_0(\Sigma_1^B,\Sigma_2^B)$ and $C_2=f(C_1)$ be the boundary curve of Σ_2^B onto which f maps C_1. Then $f\circ\phi\in\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1,C_2)$.*

2.4. **The class of non-overlapping mappings and its complex structure.** Now we recall some of the definitions and theorems from [24, 25] which will be necessary in the rest of the paper.

We define a class of non-overlapping mappings into a punctured Riemann surface. Let $\mathbb{D}_0$ denote the punctured disk $\mathbb{D}\backslash\{0\}$. Let Σ be a compact Riemann surface with punctures $p_1,\ldots,p_n$.

Definition 2.21. The class of non-overlapping quasiconformally extendible maps $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ into Σ is the set of n-tuples $(\phi_1,\ldots,\phi_n)$ where

1. For all $i\in\{1,\ldots,n\}$, $\phi_i:\mathbb{D}_0\to\Sigma$ is holomorphic, and has a quasiconformal extension to a neighborhood of $\overline{\mathbb{D}}$.
2. The continuous extension of ϕ_i takes 0 to p_i
3. For any $i\neq j$, $\overline{\phi_i(\mathbb{D})}\cap\overline{\phi_j(\mathbb{D})}$ is empty.

It was shown in [22] that $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ is a complex Banach manifold.

As in the previous section, we need to refine the class of non-overlapping mappings. We first introduce some terminology. Denote the compactification of a punctured surface Σ by $\overline{\Sigma}$.

Definition 2.22. An n-chart on Σ is a collection of open sets $E_1,\ldots,E_n$ contained in the compactification of Σ such that $E_i\cap E_j$ is empty whenever $i\neq j$, together with local parameters $\zeta_i:E_i\to\mathbb{C}$ such that $\zeta_i(p_i)=0$.

In the following, we will refer to the charts (ζ_i,E_i) as being on Σ, with the understanding that they are in fact defined on the compactification. Similarly, non-overlapping maps $(f_1,\ldots,f_n)$ will be extended by the removable singularities theorem to the compactification, without further comment.

Definition 2.23. Let $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ be the set of n-tuples of maps $(f_1,\ldots,f_n)\in\mathcal{O}^{\mathrm{qc}}(\Sigma)$ such that for any choice of n-chart $\zeta_i:E_i\to\mathbb{C}$, $i=1,\ldots,n$ satisfying $\overline{f_i(\mathbb{D})}\subset E_i$ for all $i=1,\ldots,n$, it holds that $\zeta_i\circ f_i\in\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

The space $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma)$ is well-defined, in the sense that if an n-tuple $(f_1, \ldots, f_n)$ satisfies the definition with respect to a particular n-chart, then it satisfies the definition with respect to any other n-chart satisfying the condition $\overline{f_i(\mathbb{D})} \subset E_i$ [24, 25].

The following theorem plays an essential role topologically in the Hilbert manifold structure on both $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma)$ and the WP-class Teichmüller space.

Theorem 2.24. *Let E be an open neighborhood of 0 in $\mathbb{C}$. Then the set*

$$\left\{ f \in \mathcal{O}^{\rm qc} : \overline{f(\mathbb{D})} \subset E \right\}$$

is open in $\mathcal{O}^{\rm qc}$ and the set

$$\left\{ f \in \mathcal{O}^{\rm qc}_{\rm WP} : \overline{f(\mathbb{D})} \subset E \right\}$$

is open in $\mathcal{O}^{\rm qc}_{\rm WP}$.

Composition on the left by h is holomorphic operation in both $\mathcal{O}^{\rm qc}$ and $\mathcal{O}^{\rm qc}_{\rm WP}$. This was proven in [22] in the case of $\mathcal{O}^{\rm qc}$. The corresponding theorem in the WP case is considerably more delicate [24, 25]. This fact plays an essential role in the construction of a holomorphic atlas on both $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma^P)$ and the WP-class Teichmüller space.

Theorem 2.25. *Let $K \subset \mathbb{C}$ be a compact set which is the closure of an open neighborhood K_{int} of 0 and let A be an open set in $\mathbb{C}$ containing K. If U is the open set*

$$U = \{ g \in \mathcal{O}^{\rm qc}_{\rm WP} \, : \, \overline{g(\mathbb{D})} \subset K_{int} \},$$

and $h : A \to \mathbb{C}$ is a one-to-one holomorphic map such that $h(0) = 0$, then the map $f \mapsto h \circ f$ from U to $\mathcal{O}^{\rm qc}_{\rm WP}$ is holomorphic.

Remark 2.26. The fact that U is open follows from Theorem 2.24.

Now, we define the topological structure of $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma)$.

Definition 2.27. For any n-chart $(\zeta, E) = (\zeta_1, E_1, \ldots, \zeta_n, E_n)$ (see Definition 2.22), we say that an n-tuple $U = (U_1, \ldots, U_n) \subset \mathcal{O}^{\rm qc}_{\rm WP} \times \cdots \times \mathcal{O}^{\rm qc}_{\rm WP}$, with U_i open in $\mathcal{O}^{\rm qc}_{\rm WP}$, is compatible with (ζ, E) if $\overline{f(\mathbb{D})} \subset \zeta_i(E_i)$ for all $f \in U_i$.

For any n-chart (ζ, E) and compatible open subset U of $\mathcal{O}^{\rm qc}_{\rm WP} \times \cdots \times \mathcal{O}^{\rm qc}_{\rm WP}$ let

$$\begin{aligned} V_{\zeta,E,U} &= \{ g \in \mathcal{O}^{\rm qc}_{\rm WP}(\Sigma) \, : \, \zeta_i \circ g_i \in U_i, \quad i = 1, \ldots, n \} \\ &= \{ (\zeta_1^{-1} \circ h_1, \ldots, \zeta_n^{-1} \circ h_n) \, : \, h_i \in U_i, \quad i = 1, \ldots, n \}. \end{aligned} \tag{2.3}$$

Definition 2.28 (base a for topology on $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma)$)**.** Let

$$\mathcal{V} = \{ V_{\zeta,E,U} \, : \, (\zeta, E) \text{ an } n\text{-chart}, \, U \text{ compatible with } (\zeta, E) \}.$$

Theorem 2.29. *The set $\mathcal{V}$ is the base for a topology on $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma)$. This topology is Hausdorff and second countable.*

Remark 2.30. In particular, $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma)$ is separable since it is second countable and Hausdorff.

We make one final simple but useful observation regarding the base $\mathcal{V}$.

For a Riemann surface Σ denote by $\mathcal{V}(\Sigma)$ the base for $\mathcal{O}^{\rm qc}_{\rm WP}(\Sigma)$ given in Definition 2.28. For a biholomorphism $\rho : \Sigma \to \Sigma_1$ of Riemann surfaces Σ and Σ_1, and for any $V \in \mathcal{V}(\Sigma)$, let

$$\rho(V) = \{ \rho \circ \phi \, : \, \phi \in V \}$$

and

$$\rho(\mathcal{V}(\Sigma)) = \{ \rho(V) \, : \, V \in \mathcal{V} \}.$$

Theorem 2.31. *If $\rho : \Sigma \to \Sigma_1$ is a biholomorphism between punctured Riemann surfaces Σ and Σ_1 then $\rho(\mathcal{V}(\Sigma)) = \mathcal{V}(\Sigma_1)$.*

Definition 2.32 (standard charts on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$)**.** Let (ζ, E) be an n-chart on Σ and let $\kappa_i \subset E_i$ be compact sets containing p_i. Let $K_i = \zeta_i(\kappa_i)$. Let $U_i = \{\psi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{\psi(\mathbb{D})} \subset \mathrm{interior}(K_i)\}$. Each U_i is open by Theorem 2.24 and $U = (U_1, \ldots, U_n)$ is compatible with (ζ, E) so we have $V_{\zeta,E,U} \in \mathcal{V}$. A standard chart on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is a map

$$\begin{aligned} T : V_{\zeta,E,U} &\longrightarrow \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \\ (f_1, \ldots, f_n) &\longmapsto (\zeta_1 \circ f_1, \ldots, \zeta_n \circ f_n). \end{aligned}$$

Remark 2.33. To obtain a chart into a Hilbert space, one simply composes with χ as defined by (2.2). Abusing notation somewhat and defining χ^n by

$$\begin{aligned} \chi^n \circ T : V_{\zeta,E,U} &\longrightarrow \bigoplus^n A_1^2(\mathbb{D}) \oplus \mathbb{C} \\ (f_1, \ldots, f_n) &\longmapsto (\chi \circ \zeta_1 \circ f_1, \ldots, \chi \circ \zeta_n \circ g_n) \end{aligned}$$

we obtain a chart into $\bigoplus^n A_1^2(\mathbb{D}) \oplus \mathbb{C}$. Since $\chi(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})$ is an open subset of $A_1^2(\mathbb{D}) \oplus \mathbb{C}$ by Theorem 2.3, and χ defines the complex structure $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, we may treat T as a chart with the understanding that the true charts are obtained by composing with χ^n.

Theorem 2.34. *Let Σ be a punctured Riemann surface of type (g, n). With the atlas consisting of the standard charts of Definition 2.32, $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is a complex Hilbert manifold, locally biholomorphic to $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.*

Remark 2.35 (chart simplification). Now that this theorem is proven, we can simplify the definition of the charts. For an n-chart (ζ, E), if we let $U_i = \{f \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{f(\mathbb{D})} \subset \zeta_i(E_i)\}$, then the charts T are defined on $V_{\zeta,E,U}$. It is easy to show that T is a biholomorphism on $V_{\zeta,E,U}$, since any $f \in V_{\zeta,E,U}$ is contained in some $V_{\zeta,E,W} \subset V_{\zeta,E,U}$ which satisfies Definition 2.32, and thus T is a biholomorphism on $V_{\zeta,E,W}$ by Theorem 2.34.

Remark 2.36 (standard charts on $\mathcal{O}^{\mathrm{qc}}(\Sigma)$). A standard chart on $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ is defined in the same way as Definition 2.32 and its preamble, by replacing $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ with $\mathcal{O}^{\mathrm{qc}}$ everywhere. Furthermore with this atlas $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ is a complex Banach manifold [22].

Finally, we show that the inclusion map $I : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \to \mathcal{O}^{\mathrm{qc}}$ is holomorphic.

Theorem 2.37. *The complex manifold $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ is holomorphically contained in $\mathcal{O}^{\mathrm{qc}}(\Sigma)$ in the sense that the inclusion map $I : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma) \to \mathcal{O}^{\mathrm{qc}}(\Sigma)$ is holomorphic.*

3. The rigged Teichmüller space is a Hilbert manifold

In [21], two of the authors proved that the Teichmüller space of a bordered surface is (up to a quotient by a discrete group) the same as a certain rigged Teichmüller space whose corresponding rigged moduli space appears naturally in two-dimensional conformal field theory [8, 12, 27]. We will use this fact to define a Hilbert manifold structure on the WP-class Teichmüller space in Section 4.

First we must define an atlas on rigged Teichmüller space, and this is the main task of the current section. We will achieve this by using universality of the universal Teichmüller curve together with a variational technique called *Schiffer variation* as adapted to the quasiconformal Teichmüller setting by Gardiner [9] and Nag [17, 18]. This overall approach was first developed in the thesis of the first author [20] for the case of analytic riggings.

3.1. **Definition of rigged Teichmüller space.** We first recall the definition of the usual Teichmüller space. The reader is referred to Section 2.2 for terminology regarding Riemann surfaces.

Definition 3.1. Fix a Riemann surface X (of any topological type). Let

$$T(X) = \{(X, f, X_1)\}/\sim$$

where

(1) X_1 is a Riemann surface of the same topological type as X.
(2) $f : X \to X_1$ is a quasiconformal homeomorphism (the *marking map*).
(3) the equivalence relation ($\sim$) is defined by $(X, f_1, X_1) \sim (X, f_2, X_2)$ if and only if there exists a biholomorphism $\sigma : X_1 \to X_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary.

The term *rel boundary* means that the homotopy is the identity on the boundary throughout the homotopy.

It is a standard fact of Teichmüller theory (see for example [18]) that if X is a punctured surface of type (g, n) then $T(X)$ is a complex manifold of dimension $3g - 3 + n$, and if X is a bordered surface of type (g, n) then $T(X)$ is an infinite-dimensional complex Banach manifold.

Using the set $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ we now define the WP-*class rigged Teichmüller space*, denoted by $\widetilde{T}_{\mathrm{WP}}(\Sigma)$.

Definition 3.2. Fix a punctured Riemann surface of type (g, n). Let

$$\widetilde{T}_{\mathrm{WP}}(\Sigma) = \{(\Sigma, f, \Sigma_1, \phi)\}/\sim$$

where

(1) Σ_1 is a punctured Riemann surface of type (g, n)
(2) $f : \Sigma \to \Sigma_1$ is a quasiconformal homeomorphism
(3) $\phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1)$.
(4) Two quadruples are said to be equivalent, denoted by $(\Sigma, f_1, \Sigma_1, \phi_1) \sim (\Sigma, f_2, \Sigma_2, \phi_2)$, if and only if there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary and $\phi_2 = \sigma \circ \phi_1$.

The equivalence class of $(\Sigma, f_1, \Sigma_1, \phi_1)$ will be denoted $[\Sigma, f_1, \Sigma_1, \phi_1]$

Condition (2) can be stated in two alternate ways. One is to require that f maps the compactification of Σ into the compactification of Σ_1, and takes the punctures of Σ to the punctures of Σ_1 (now thought of as marked points). The other is to say simply that f is a quasiconformal map between Σ and Σ_1. Since f is quasiconformal its extension to the compactification will take punctures to punctures. Thus condition (2) does not explicitly mention the punctures.

In [21], two of the authors defined a rigged Teichmüller space $\widetilde{T}(\Sigma)$ with $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1)$ replaced by $\mathcal{O}^{\mathrm{qc}}(\Sigma_1)$ in the above definition. It was demonstrated in [21] that $\widetilde{T}(\Sigma)$ has a complex Banach manifold structure, which comes from the fact that it is a quotient of the Teichmüller space of a bordered surface by a properly discontinuous, fixed-point free group of biholomorphisms. In [23] they demonstrated that it is fibered over $T(\Sigma)$, where the fiber over a point $[\Sigma, f_1, \Sigma_1]$ is biholomorphic to $\mathcal{O}^{\mathrm{qc}}(\Sigma_1)$. Furthermore, the complex structure of $\mathcal{O}^{\mathrm{qc}}(\Sigma_1)$ is compatible with the complex structure that the fibers inherit from $\widetilde{T}(\Sigma)$.

This notion of a *rigged Teichmüller space* was first defined, in the case of analytic riggings, by one of the authors in [20], and it was used to obtain a complex Banach manifold structure on the analytically rigged moduli space. However, in the case of analytic riggings the connection to the complex structure of the infinite-dimensional Teichmüller space of bordered surfaces can not be made.

From now on, any punctured Riemann surface is assumed to satisfy $2g + 2 - n > 0$. We would now like to demonstrate that $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ has a natural complex Hilbert manifold structure which arises from $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$, and that this also passes to the rigged Riemann moduli space. In Section 4, we will use it to construct a complex Hilbert manifold structure on a WP-class Teichmüller space of a bordered surface. To accomplish these tasks we use a natural coordinate system developed in [20, 23], which is based on Gardiner-Schiffer variation and the complex structure on $\mathcal{O}^{\mathrm{qc}}(\Sigma)$. We will refine these coordinates to $\widetilde{T}_{\mathrm{WP}}(\Sigma)$.

We end this section with a basic result concerning the above definition. Since Σ satisfies $2g + 2 - n > 0$ we have the following well known theorem [18].

Theorem 3.3. *If $\sigma : \Sigma \to \Sigma$ is a biholomorphism that is homotopic to the identity then σ is the identity.*

Corollary 3.4. *If $[\Sigma, f_1, \Sigma_1, \phi_1] = [\Sigma, f_1, \Sigma_1, \phi_2] \in \widetilde{T}_{\mathrm{WP}}(\Sigma)$ then $\phi_1 = \phi_2$.*

3.2. **Marked families.** In this section we collect some standard definitions and facts about marked holomorphic families of Riemann surfaces and the universality of the Teichmüller curve. These will play a key role in the construction of an atlas on rigged Teichmüller space. A full treatment appears in [6], and also in the books [14, 18].

Definition 3.5. A holomorphic family of complex manifolds is a pair of connected complex manifolds (E, B) together with a surjective holomorphic map $\pi : E \to B$ such that (1) π is topologically a locally trivial fiber bundle, and (2) π is a split submersion (that is, the derivative is a surjective map whose kernel is a direct summand).

Definition 3.6. A *morphism of holomorphic families* from (E', B') and (E, B) is a pair of holomorphic maps (α, β) with $\alpha : B' \to B$ and $\beta : E' \to E$ such that

$$\begin{array}{ccc} E' & \xrightarrow{\beta} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\alpha} & B \end{array}$$

commutes, and for each fixed $t \in B'$, the restriction of β to the fiber $\pi'^{-1}(t)$ is a biholomorphism onto $\pi^{-1}(\alpha(t))$.

Throughout, (E, B) will be a holomorphic family of Riemann surfaces; that is, each fiber $\pi^{-1}(t)$ is a Riemann surface. Moreover, since our trivialization will always be global we specialize the standard definitions (see [6]) to this case in what follows.

Let Σ be a punctured Riemann surface of type (g, n). This fixed surface Σ will serve as a model of the fiber.

Definition 3.7.

(1) A *global trivialization* of (E, B) is a homeomorphism $\theta : B \times \Sigma \to E$ such that $\pi(\theta(t, x)) = t$ for all $(t, x) \in B \times \Sigma$.

(2) A global trivialization θ is a *strong trivialization* if for fixed $x \in \Sigma$, $t \mapsto \theta(t,x)$ is holomorphic, and for each $t \in B$, $x \mapsto \theta(t,x)$ is a quasiconformal map from Σ onto $\pi^{-1}(t)$.
(3) $\theta : B \times \Sigma \to E$ and $\theta' : B \times \Sigma \to E$ are *compatible* if and only if $\theta'(t,x) = \theta(t, \phi(t,x))$ where for each fixed t, $\phi(t,x) : \Sigma \to \Sigma$ is a quasiconformal homeomorphism that is homotopic to the identity rel boundary.
(4) A *marking* $\mathcal{M}$ for $\pi : E \to B$ is an equivalence class of compatible strong trivializations.
(5) A *marked family of Riemann surfaces* is a holomorphic family of Riemann surfaces with a specified marking.

Remark 3.8. Let θ and θ' be compatible strong trivializations. For each fixed $t \in B$, $[\Sigma, \theta(t,\cdot), \pi^{-1}(t)] = [\Sigma, \theta'(t,\cdot), \pi^{-1}(t)]$ in $T(\Sigma)$ (see Definition 3.1). So a marking specifies a Teichmüller equivalence class for each t.

We now define the equivalence of marked families.

Definition 3.9. A *morphism of marked families* from $\pi' : E' \to B'$ to $\pi : E \to B$ is a pair of holomorphic maps (α, β) with $\beta : E' \to E$ and $\alpha : B' \to B$ such that

(1) (α, β) is a morphism of holomorphic families, and
(2) the markings $B' \times \Sigma \to E$ given by $\beta(\theta'(t,x))$ and $\theta(\alpha(t), x)$ are compatible.

The second condition says that (α, β) preserves the marking.

Remark 3.10 (relation to Teichmüller equivalence). Define $E = \{(s, Y_s)\}_{s \in B}$ and $E' = \{(t, X_t)\}_{t \in B'}$ to be marked families of Riemann surfaces with markings $\theta(s,x) = (s, g_s(x))$ and $\theta'(t,x) = (t, f_t(x))$ respectively. Say (α, β) is a morphism of marked families, and define σ_t by $\beta(t,y) = (\alpha(t), \sigma_t(y))$. Then $\beta(\theta'(t,x)) = (\alpha(t), \sigma_t(f_t(x)))$ and $\theta(\alpha(t), x) = (\alpha(t), g_{\alpha(t)}(x))$. The condition that (α, β) is a morphism of marked families is simply that $\sigma_t \circ f_t$ is homotopic rel boundary to $g_{\alpha(t)}$. That is, when $s = \alpha(t)$, $[\Sigma, f_t, X_t] = [\Sigma, g_s, Y_s]$ via the biholomorphism $\sigma_t : X_t \to Y_s$.

The universal Teichmüller curve, denoted by $\pi_T : \mathcal{T}(\Sigma) \to T(\Sigma)$, is a marked holomorphic family of Riemann surfaces with fiber model Σ. The following universal property of $\mathcal{T}(\Sigma)$ (see [6, 14, 18]) is all that we need for our purposes.

Theorem 3.11 (Universality of the Teichmüller curve)**.** *Let $\pi : E \to B$ be a marked holomorphic family of Riemann surfaces with fiber model Σ of type (g,n) with $2g-2+n > 0$, and trivialization θ. Then there exists a unique map (α, β) of marked families from $\pi : E \to B$ to $\pi_T : \mathcal{T}(\Sigma) \to T(\Sigma)$. Moreover, the canonical "classifying" map $\alpha : B \to T(\Sigma)$ is given by $\alpha(t) = [\Sigma, \theta(t,\cdot), \pi^{-1}(t)]$.*

3.3. **Schiffer variation.** The use of Schiffer variation to construct analytic coordinates on Teichmüller space by using quasiconformal deformations is due to Gardiner [9] and Nag [17, 18]. We review the construction in some detail, as it will be used in a crucial way in the subsequent sections.

Let $B_R = \{z \in \mathbb{C} : |z| < R\}$, and for $r < R$ let $\mathbb{A}(r,R) = \{z \in \mathbb{C} : r < |z| < R\}$ as before. Choose r and R such that $0 < r < 1 < R$. Let Σ be a (possibly punctured) Riemann surface and $\xi : U \to \mathbb{C}$ be local holomorphic coordinate on an open connected set $U \subset \Sigma$ such that $B_R \subset \text{Image}(\xi)$. Let $D = \xi^{-1}(\mathbb{D})$, which we call a parametric disk.

Define $v^\epsilon : \mathbb{A}(r,R) \longrightarrow \mathbb{C}$ by $v^\epsilon(z) = z + \epsilon/z$ where $\epsilon \in \mathbb{C}$. For $|\epsilon|$ sufficiently small v is a biholomorphism onto its image. Let D^ϵ be the interior of the analytic Jordan curve $v^\epsilon(\partial\mathbb{D})$. We regard $\overline{D^\epsilon}$ as a bordered Riemann surface (with the standard complex structure inherited from $\mathbb{C}$) with analytic boundary parametrization given by $v^\epsilon : \mathbb{S}^1 \to \partial D^\epsilon$. We also have the Riemann surface $\Sigma \setminus D$ with the boundary analytically parametrized by $\xi^{-1}|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial(\Sigma \setminus D)$.

We now sew $\overline{D^\epsilon}$ and $\Sigma \setminus D$ along their boundaries by identifying $x \in \partial(\Sigma \setminus D)$ with $x' \in \partial\overline{D^\epsilon}$ if and only if $x' = (v^\epsilon \circ \xi)(x)$. Let

$$\Sigma^\epsilon = (\Sigma \setminus D) \sqcup \overline{D^\epsilon} \ / \text{ boundary identification}$$

and we say this Riemann surface is obtained from Σ by *Schiffer variation* of complex structure on D. Let

$$\iota^\epsilon : \Sigma \setminus D \longrightarrow \Sigma^\epsilon \quad \text{and} \quad \iota^\epsilon_D : D^\epsilon \to \Sigma^\epsilon$$

be the holomorphic inclusion maps. With a slight abuse of notation we could use the identity map in place of ι^ϵ, however the extra notation will make the following exposition clearer.

Define $w^\epsilon : \overline{\mathbb{D}} \to \overline{D^\epsilon}$ by $w^\epsilon(z) = z + \epsilon\bar{z}$. Note that w^ϵ is a homeomorphism, and on the boundary $v^\epsilon = w^\epsilon$. Define the quasiconformal homeomorphism $\nu^\epsilon : \Sigma \to \Sigma^\epsilon$ by

$$\nu^\epsilon(x) = \begin{cases} \iota^\epsilon(x), & x \in \Sigma \setminus D \\ (\iota^\epsilon_D \circ w \circ \xi)(x), & x \in D. \end{cases}$$

So we now have a point $[\Sigma, \nu^\epsilon, \Sigma^\epsilon] \in T(\Sigma)$ obtained by Schiffer variation of the base point $[\Sigma, \mathrm{id}, \Sigma]$.

To get coordinates on $T(\Sigma)$ we perform Schiffer variation on d disks where $d = 3g - 3 + n$ is the complex dimension of $T(\Sigma)$. Let $(D_1, \ldots, D_d)$ be d disjoint parametric disks on Σ, where $D_i = (\xi_i)^{-1}(\mathbb{D})$ for suitably chosen local coordinates ξ_i. Let $D = D_1 \cup \cdots \cup D_d$ and let $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \mathbb{C}^d$. Schiffer variation can be performed on the d disks to get a new surface which we again denote by Σ^ϵ. The map ν^ϵ becomes

$$\nu^\epsilon(x) = \begin{cases} \iota^\epsilon(x), & x \in \Sigma \setminus D \\ (\iota^\epsilon_D \circ w^{\epsilon_i} \circ \xi_i)(x), & x \in D_i \ , i = 1, \ldots, d. \end{cases} \tag{3.1}$$

The following theorem is the main result on Schiffer variation [9, 18]. Let $\Omega \subset \mathbb{C}^d$ be an open neighborhood of 0 such that Schiffer variation is defined for $\epsilon \in \Omega$. Define

$$\begin{aligned} \mathcal{S} : \Omega &\longrightarrow T(\Sigma) \\ \epsilon &\longmapsto [\Sigma, \nu^\epsilon, \Sigma^\epsilon]. \end{aligned} \tag{3.2}$$

Theorem 3.12. *Given any d disjoint parametric disks on Σ, it is possible to choose the local parameters ξ_i such that on some open neighborhood Ω of $0 \in \mathbb{C}^n$, $\mathcal{S} : \Omega \to \mathcal{S}(\Omega)$ is a biholomorphism. That is, the parameters $(\epsilon_1, \ldots, \epsilon_d)$ provide local holomorphic coordinates for Teichmüller space in a neighborhood of $[\Sigma, id, \Sigma]$*

It is important to note that we are free to choose the domains D_i on which the Schiffer variation is performed.

By a standard change of base point argument we can use Schiffer variation to produce a neighborhood of any point $[\Sigma, f, \Sigma_1] \in T(\Sigma)$. Performing Schiffer variation on Σ_1 gives a neighborhood $\mathcal{S}(\Omega)$ of $[\Sigma_1, \mathrm{id}, \Sigma_1] \in T(\Sigma_1)$. Consider the change of base point biholomorphism (see [18, Sections 2.3.1 and 3.2.5]) $f^* : T(\Sigma_1) \to T(\Sigma)$ given by $f^*([\Sigma_1, g, \Sigma_2] =$

$[\Sigma, g \circ f, \Sigma_2]$. Then $f^* \circ \mathcal{S}$ is a biholomorphism onto its image $f^*(\mathcal{S}(\Omega)) = \{[\Sigma, \nu^\epsilon \circ f, \Sigma_1^\epsilon]\}$ which is a neighborhood of $[\Sigma, f, \Sigma_1] \in T(\Sigma)$.

Thus, denoting $f^* \circ \mathcal{S}$ itself by $\mathcal{S}$, the Schiffer variation

$$\begin{aligned} \mathcal{S} : \Omega &\longrightarrow T(\Sigma) \\ \epsilon &\longmapsto [\Sigma, \nu^\epsilon \circ f, \Sigma^\epsilon]. \end{aligned} \tag{3.3}$$

produces a neighborhood of $[\Sigma, f, \Sigma_1] \in T(\Sigma)$.

3.4. **Marked Schiffer family.** Fix a point $[\Sigma, f, \Sigma_1] \in T(\Sigma)$. We will show that Schiffer variation on Σ_1 produces a marked holomorphic family of Riemann surfaces with fiber Σ_1^ϵ over the point ϵ and marking $\nu^\epsilon \circ f$. Since this construction does not appear in the literature, we present it here in some detail as it is an essential ingredient in our later proofs. An efficient way to describe the family is to do the sewing for all ϵ simultaneously.

For $i = 1, \dots, d$, let Ω_i be connected open neighborhoods of $0 \in \mathbb{C}$ such that $\Omega = \Omega_1 \times \cdots \times \Omega_d$ is an open subset of $\mathbb{C}^d$ for which Schiffer variation is defined and Theorem 3.12 implies that $\mathcal{S} : \Omega \to \mathcal{S}(\Omega) \subset T(\Sigma)$ is a biholomorphism.

Define, for each $i = 1, \dots, d$,

$$\begin{aligned} w_i : \Omega_i \times \mathbb{D} &\longrightarrow \mathbb{C} \times \mathbb{C} \\ (\epsilon_i, z) &\longmapsto (\epsilon_i, w^{\epsilon_i}(z)), \end{aligned}$$

$$\begin{aligned} v_i : \Omega_i \times A_r^1 &\longrightarrow \mathbb{C} \times \mathbb{C} \\ (\epsilon_i, z) &\longmapsto (\epsilon_i, v^{\epsilon_i}(z)), \end{aligned}$$

and let

$$Y_i = w_i(\Omega_i \times \mathbb{D}).$$

Since w_i is a homeomorphism, Y_i is open and so inherits a complex manifold structure from $\mathbb{C} \times \mathbb{C}$. Note that for fixed ϵ_i, $\{z \,|\, (\epsilon_i, z) \in Y_i\} = D^{\epsilon_i}$.

With $r < 1$ as in the construction of Schiffer variation, let $D_i^r = \xi_i^{-1}(B(0, r))$ and $D^r = D_1^r \cup \cdots \cup D_d^r$. Let

$$X = \Omega \times (\Sigma_1 \setminus \overline{D^r})$$

and endow it with the product complex manifold structure. Define the map

$$\begin{aligned} \rho_i : \Omega \times (D_i \setminus \overline{D_i^r}) &\longrightarrow v(\Omega \times A_r^1) \\ (\epsilon_i, x) &\longmapsto (\epsilon, v^{\epsilon_i}(\xi_i(x)). \end{aligned}$$

From the definition of v^{ϵ_i} it follows directly that ρ_i is a biholomorphism from an open subset of X to an open subset of Y_i.

Using the standard gluing procedure for complex manifolds (see for example [7, page 170]) we can make the following definition.

Definition 3.13. Let $S(\Omega, D)$ be the complex manifold obtained by gluing X to $Y_1, \dots, Y_d$ using the biholomorphisms $\rho_1, \dots, \rho_d$.

The inclusions $\iota_X : X \hookrightarrow S(\Omega, D)$ and $\iota_{Y_i} : Y_i \hookrightarrow S(\Omega, D)$ are holomorphic. Moreover, since r just determines the size of the overlap, $S(\Omega, D)$ is independent of r.

Equivalently, we can think of gluing $\Omega \times (\Sigma_1 \setminus D)$ and $w(\Omega \times \overline{\mathbb{D}})$ using the ρ_i restricted to $\Omega \times \partial D_i$ to identify the boundary components. For each fixed ϵ this gluing is precisely that used to define Σ_1^ϵ. So we see that

$$S(\Omega, D) = \{(\epsilon, x) : \epsilon \in \Omega, x \in \Sigma_1^\epsilon\}.$$

Define the projection map

$$\begin{aligned} \pi_S : S(\Omega, D) &\longrightarrow \Omega \\ (\epsilon, x) &\longmapsto \epsilon \end{aligned}$$

and the trivialization

$$\begin{aligned} \theta : \Omega \times \Sigma &\longrightarrow S(\Omega, D) \\ (\epsilon, x) &\longmapsto (\epsilon, (\nu^\epsilon \circ f)(x)). \end{aligned} \tag{3.4}$$

It is immediate that π_S is onto, holomorphic and defines a topologically trivial bundle.

Definition 3.14. We call $\pi_S : S(\Omega, D) \to \Omega$ with trivialization θ a marked Schiffer family.

We will have use for explicit charts on $S(\Omega, D)$, but only on the part that is disjoint from the Schiffer variation. Let (U, ζ) be a chart on $\Sigma_1 \setminus \overline{D}$. Recall that $\iota^\epsilon : \Sigma_1 \setminus \overline{D} \to \Sigma_1^\epsilon$ is the holomorphic inclusion map. Let

$$\tilde{U} = \{(\epsilon, x) \,|\, \epsilon \in \Omega, x \in \iota^\epsilon(U)\} \subset S(\Omega, D) \tag{3.5}$$

and define

$$\begin{aligned} \tilde{\zeta} : \tilde{U} &\longrightarrow \mathbb{C} \times \mathbb{C} \\ (\epsilon, x) &\longmapsto \big(\epsilon, (\zeta \circ (\iota^\epsilon)^{-1})(x)\big) \end{aligned} \tag{3.6}$$

Then $(\tilde{\zeta}, \tilde{U})$ is a holomorphic chart on $S(\Omega, D)$.

Note that with a slight of abuse of notation we could simply write $\tilde{U} = \Omega \times U$ and define $\tilde{\zeta}$ by $(\epsilon, x) \mapsto (\epsilon, \zeta(x))$, but we will refrain from doing so.

Theorem 3.15. *A marked Schiffer family is a marked holomorphic family of Riemann surfaces.*

Proof. We must check the conditions in Definitions 3.5 and 3.7.

Because ν^ϵ is a quasiconformal homeomorphism, $\theta(\epsilon, z)$ is a homeomorphism, and for fixed ϵ, $\theta(\epsilon, z)$ is quasiconformal. Next, we show that for fixed x, $\theta(\epsilon, x)$ is holomorphic in ϵ.

(1) If $x \in \Sigma \setminus f^{-1}(D_i)$ then $\theta(\epsilon, x) \in \iota_X(X)$. Let ζ and ζ' be a local coordinates in neighborhoods of x and $f(x)$ respectively, and let $z = \zeta(x)$. Use these to form the product charts on $\Omega \times \Sigma$ and X. From the definition of ν^ϵ (see (3.1)) it follows directly that in terms of local coordinates $\theta(\epsilon, x)$ is the map $(\epsilon, z) \mapsto (\epsilon, (\zeta' \circ f \circ \zeta^{-1}(z))$. Since the second entry is independent of ϵ the map is clearly holomorphic in ϵ.

(2) If $x \in f^{-1}(D_i)$ then $\theta(\epsilon, x) \in \iota_{Y_i}(Y_i)$. Let η be a coordinate map on $f^{-1}(D_i)$ and let $z = \eta(x)$. Use $(\epsilon, t) \mapsto (\epsilon, \zeta(t))$ as the product chart on $\Omega \times f^{-1}(D_i)$. Let $y = \xi_i \circ f \circ \eta^{-1}(z)$ which is independent of ϵ. Then in terms of local coordinates, θ becomes $(\epsilon, z) \mapsto (\epsilon, w^{\epsilon_i}(y))$. Since $w^{\epsilon_i}(y) = y + \epsilon_i \bar{y}$, it is certainly holomorphic in ϵ for fixed y.

Conditions (1) and (2) of Definition 3.7 are thus satisfied. It remains to prove condition (2) of Definition 3.5.

Because $\theta(\epsilon, x)$ is holomorphic in ϵ, $S(\Omega, D)$ has a holomorphic section though every point. This implies that $\pi_S : S(\Omega, D) \to \Omega$ is a holomorphic split submersion (see for example [18, section 1.6.2], and also [14, Section 6.2] for an alternate definition of marked families).

So $\theta(\epsilon, z)$ is a strong trivialization and hence $S(\Omega, D)$ is a marked family of Riemann surfaces. □

We will need the following lemma regarding maps between marked Schiffer families. We consider two Schiffer families, whose corresponding neighborhoods in Teichmüller space intersect on an open se,t and the morphism between these families.

For $i = 1, 2$, let $\pi_1 : S_i(\Omega_i, D_i) \to \Omega_i$ be marked Schiffer families based at $[\Sigma, f_i, \Sigma_i]$. Let $\mathcal{S}_i : \Omega_i \to T(\Sigma)$ be the corresponding variation maps defined by (3.3), and assume that $\mathcal{S}_1(\Omega_1) \cap \mathcal{S}_2(\Omega_2)$ is non-empty. Let N be any connected component of $\mathcal{S}_1(\Omega_1) \cap \mathcal{S}_2(\Omega_2)$, and let $\Omega_i' = \mathcal{S}_i^{-1}(N)$.

Consider the marked Schiffer families $S_i(\Omega_i', D_i) = \pi_i^{-1}(\Omega_i')$ with trivializations $\theta_i : \Omega_i' \times \Sigma \to S_i(\Omega_i', D_i)$ defined by $\theta_i(\epsilon, x) = (\epsilon, (\nu_i^\epsilon \circ f_i)(x))$. For ease of notation we write $S_i' = S_i(\Omega_i', D_i)$.

Recall that throughout we are assuming that Σ is of type (g, n) with $2g - 2 + n > 0$.

Lemma 3.16. *There is a unique invertible morphism of marked families (α, β) from $\pi_1 : S_1' \to \Omega_1'$ to $\pi_2 : S_2' \to \Omega_2'$. In particular, the following hold:*

1. *There is a unique map $\alpha : \Omega_1' \to \Omega_2'$ such that $[\Sigma, \nu_1^\epsilon \circ f_1, \Sigma_1^\epsilon] = [\Sigma, \nu_2^{\alpha(\epsilon)} \circ f_2, \Sigma_2^{\alpha(\epsilon)}]$, and α is a biholomorphism.*
2. *For each $\epsilon \in \Omega_1'$, there is a unique biholomorphism $\sigma_\epsilon : \Sigma_1^\epsilon \to \Sigma_2^{\alpha(\epsilon)}$ realizing the equivalence in (1).*
3. *The function $\beta(\epsilon, z) = (\alpha(\epsilon), \sigma_\epsilon(z))$ is a biholomorphism on $\pi_1^{-1}(\Omega_1') \subset S_1(\Omega_1, D_1)$.*

Proof. By Theorem 3.11 there are unique mappings of marked families (α_i, β_i) from $\pi_i : S_i' \to \Omega_i'$ to $\pi_T : \mathcal{T}(\Sigma) \to T(\Sigma)$. By Theorem 3.12 and the fact that $\alpha_i = \mathcal{S}_i$ from equation (3.3) we see that α_i is injective. Since β_i is injective fiberwise and $\alpha_i \circ \pi_i = \pi_T \circ \beta_i$ it follows that β_i is injective. So α_i and β_i are biholomorphisms onto their images since they are holomorphic and injective functions on finite-dimensional complex spaces.

Let $\alpha = \alpha_2^{-1} \circ \alpha_1$ and $\beta = \beta_2^{-1} \circ \beta_1$; these are biholomorphisms from $\Omega_1' \to \Omega_2'$ and $S_1' \to S_2'$ respectively. Then (α, β) is the unique map of marked families from $\pi_1 : S_1' \to \Omega_1'$ to $\pi_2 : S_2' \to \Omega_2'$, and has inverse $(\alpha^{-1}, \beta^{-1})$.

The proof of (1) is completed by noting that the equation

$$[\Sigma, \nu_1^\epsilon \circ f_1, \Sigma_1^\epsilon] = [\Sigma, \nu_2^{\alpha(\epsilon)} \circ f_2, \Sigma_2^{\alpha(\epsilon)}]$$

is precisely $\alpha_1(\epsilon) = \alpha_2(\alpha(\epsilon))$, which is true by the definition of α.

Because β restricted to the fibers is a biholomorphism and $\alpha_1 \circ \pi_1 = \pi_2 \circ \beta$ we can write (as in Remark 3.10) β in the form

$$\beta(\epsilon, x) = (\alpha(\epsilon), \sigma_\epsilon(x))$$

where $\sigma_\epsilon : \Sigma_1^\epsilon \to \Sigma_2^{\alpha(\epsilon)}$ is a biholomorphism.

Since $2g - 2 + n > 0$, the uniqueness in (2) follows directly from Theorem 3.3.

We have already proved that $\beta : S_1' \to S_2'$ is a biholomorphism and so (3) is proved. □

Remark 3.17. Part (3) of the above lemma is the reason for introducing the theory of marked families. Without this theory, it is impossible to prove (or even formulate the notion of) holomorphicity in ϵ of the map σ_ϵ realizing the Teichmüller equivalence. The holomorphicity in ϵ is necessary for the proof that the transition functions on the rigged Teichmüller space are biholomorphisms (Theorem 3.27 below).

3.5. **Topology and atlas for the rigged Teichmüller space.** We will now give the rigged Teichmüller space a Hilbert manifold structure.

We begin by defining a base for the topology. Let Σ be a punctured Riemann surface of type (g, n). We fix a point $[\Sigma, f, \Sigma_1] \in T(\Sigma)$. Let (ζ, E) be an n-chart on Σ_1, let $U \subset \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ be compatible with (ζ, E), and let $V = V_{\zeta,E,U}$ (defined in equation (2.3)).

Definition 3.18. We say that a marked Schiffer family $S(\Omega, D)$ is compatible with an n-chart (ζ, E) if the closure of each disk D_i is disjoint from the closure of E_j for all i and j.

For any punctured Riemann surface Σ' denote by $\mathcal{V}(\Sigma')$ the basis of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma')$ as in Definition 2.28.

Lemma 3.19. *Let $S(\Omega, D)$ be a marked Schiffer family based at $[\Sigma, f, \Sigma_1]$ and let $V \in \mathcal{V}(\Sigma_1)$. If $S(\Omega, D)$ is compatible with V then $\nu^\epsilon(V) = \{\nu^\epsilon \circ \phi \,:\, \phi \in V\}$ is an element of $\mathcal{V}(\Sigma_1^\epsilon)$.*

Proof. Writing V in terms of its corresponding n-chart (ζ, E) and $W \subset \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \times \cdots \times \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$, this is an immediate consequence of the fact that ν^ϵ is holomorphic on the sets E_i. □

Define the set

$$F(V, S, \Delta) = \{[\Sigma, \nu^\epsilon \circ f, \Sigma_1^\epsilon, \phi] \,:\, \epsilon \in \Delta, \phi \in \nu^\epsilon(V)\}$$

where $V \in \mathcal{V}$, $S = S(\Omega, D)$ is a Schiffer variation compatible with V, and Δ is a connected open subset of Ω. The base $\mathcal{F}$ consists of such sets.

Definition 3.20. The base for the topology of $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ is

$$\mathcal{F} = \{F(V, S, \Delta) \,:\, S(\Omega, D) \text{ compatible with } V,\ \Delta \subseteq \Omega \text{ open and connected}\}.$$

It is an immediate consequence of the definition that the restriction of any $F \in \mathcal{F}$ to a fiber is open in in the following sense.

Lemma 3.21. *Let Σ and $\mathcal{F}$ be as above. For any $F \in \mathcal{F}$ and representative (Σ, f_1, Σ_1) of any point $[\Sigma, f_1, \Sigma_1] \in T(\Sigma)$*

$$\{\phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1) \,:\, [\Sigma, f_1, \Sigma_1, \phi] \in F\}$$

is an open subset of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1)$.

Proof. This follows immediately from Lemma 3.19. □

It is necessary to show that $\mathcal{F}$ is indeed a base. This will be accomplished in several steps, together with the proof that the overlap maps of the charts are biholomorphisms. The charts are given in the following definition.

Definition 3.22. For each open set $F(V,S,\Delta) \subset \widetilde{T}_{\mathrm{WP}}(\Sigma)$ we define the chart

$$\begin{aligned} G : \Delta \times U &\longrightarrow F(V,S,\Delta) \\ (\epsilon, \psi) &\longmapsto [\Sigma, \nu^\epsilon \circ f_1, \Sigma_1^\epsilon, \nu^\epsilon \circ \zeta^{-1} \circ \psi]. \end{aligned} \tag{3.7}$$

where $U \subset (\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})^n$ is related to V as in Definition 2.28 and $S = S(\Omega, D)$ is compatible with V.

Lemma 3.23. *The map G is a bijection.*

Proof. If $G(\epsilon_1, \psi_1) = G(\epsilon_1, \psi_1)$ then $\epsilon_1 = \epsilon_2$ by Theorem 3.12. Because $2g - 2 + n > 0$, Corollary 3.4 implies $\psi_1 = \psi_2$. This proves injectivity. Surjectivity of G follows from the definition of $F(V,S,\Delta)$. □

It was shown in [23], that if in the above map $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ and $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ are replaced by $\mathcal{O}^{\mathrm{qc}}$ and $\mathcal{O}^{\mathrm{qc}}(\Sigma)$, and the corresponding changes are made to the sets U_i and V_i, then these coordinates can be used to form an atlas on $\widetilde{T}^P(\Sigma)$. We need to show the same result in the WP-class setting.

Remark 3.24. Between here and the end of the proof of Lemma 3.25, we will suppress the subscripts on n-charts (ζ_i, E_i) and elements of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1)$ to avoid clutter. The subscripts which remain will distinguish n-charts on different Riemann surfaces.

When clarification is necessary we will use the notation, for example $(\zeta_{i,j},\, E_{i,j})$, where the first index labels the Riemann surface and the second labels the puncture.

We proceed as follows. We first prove two lemmas, whose purpose is to show that in a neighborhood of any point, the transition functions are defined and holomorphic on some open set. Once this is established, we show that $\mathcal{F}$ is a base, the topology is Hausdorff and separable, and the charts form a holomorphic atlas.

Some notation is necessary regarding the transition functions. Fix two points $[\Sigma, f_1, \Sigma_1]$ and $[\Sigma, f_2, \Sigma_2]$ in $T(\Sigma)$. Let G_1 and G_2 be two corresponding parametrizations as in (3.7) above, defined on $\Delta_1 \times U_1$ and $\Delta_2 \times U_2$ respectively and using the two Schiffer families $S_1(\Delta_1, D_1)$ and $S_2(\Delta_2, D_2)$. We assume that the intersection $G_1(\Delta_1 \times U_1) \cap G_1(\Delta_2 \times U_2)$ is non-empty. From the definitions of $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ and $\mathcal{S}$ it follows that $\mathcal{S}(\Delta_1) \cap \mathcal{S}(\Delta_2)$ is also non-empty. We follow the notation and setup of Lemma 3.16 and the paragraph immediately preceding it, with $\Delta_i' = \mathcal{S}_i^{-1}(N)$ replacing Ω_i', where N is any connected component of $\mathcal{S}(\Delta_1) \cap \mathcal{S}(\Delta_2)$.

Recall that in $\widetilde{T}^P_{\mathrm{WP}}(\Sigma)$, $[\Sigma, g_1, \Sigma_1, \phi_1] = [\Sigma, g_2, \Sigma_2, \phi_2]$ if and only if $[\Sigma, g_1, \Sigma_1] = [\Sigma, g_2, \Sigma_2]$ via the biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ and $\sigma \circ \phi_1 = \phi_2$. Lemma 3.16 now implies that $G_1(\epsilon, \psi) = G_2(\epsilon', \psi')$ if and only if $\epsilon' = \alpha(\epsilon)$ and

$$\nu_2^{\alpha(\epsilon)} \circ \zeta_2^{-1} \circ \psi' = \sigma_\epsilon \circ \nu_1^\epsilon \circ \zeta_1^{-1} \circ \psi.$$

Let

$$\mathcal{H}(\epsilon, z) = \mathcal{H}_\epsilon(z) = \left(\zeta_2 \circ (\nu_2^{\alpha(\epsilon)})^{-1} \circ \sigma_\epsilon \circ \nu_1^\epsilon \circ \zeta_1^{-1} \right)(z) \tag{3.8}$$

which is a function of two complex variables. We also define

$$\mathcal{G}(\epsilon, z) = (\alpha(\epsilon), \mathcal{H}(\epsilon, z)).$$

Note that this is shorthand for a collection of maps $\mathcal{H}^j(\epsilon, z)$ and $\mathcal{G}^j(\epsilon, z)$, $j = 1, \ldots, n$, where j indexes the punctures (cf. Remark 3.24). Define further

$$H : \Omega_1' \times (\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})^n \longrightarrow (\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})^n \qquad (3.9)$$
$$(\epsilon, \psi) \longmapsto \mathcal{H}_\epsilon \circ \psi.$$

The overlap maps can then be written

$$(G_2^{-1} \circ G_1)(\epsilon, \psi) = (\alpha(\epsilon), \mathcal{H}_\epsilon \circ \psi) = (\alpha(\epsilon), H(\epsilon, \psi)). \qquad (3.10)$$

Lemma 3.25. *Let $[\Sigma, f_1, \Sigma_1]$ and $[\Sigma, f_2, \Sigma_2] \in \widetilde{T}^P_{\mathrm{WP}}(\Sigma)$ for a punctured Riemann surface Σ. For $i = 1, 2$ let $\mathcal{V}_i$ be the base for the topology on $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_i)$ as in Definition 2.28. Again for $i = 1, 2$ let (ζ_i, E_i) be n-charts on Σ_i, let $V_i \in \mathcal{V}_i$ be compatible with the n-charts (ζ_i, E_i), and let $S_i(\Omega_i, D_i)$ be Schiffer variations compatible with V_i. Finally, for open connected sets $\Delta_i \subseteq \Omega_i$ consider the sets $F(V_i, S_1, \Delta_i)$ which we assume have non-empty intersection.*

Choose any $e_1 \in \Delta_1$ and $\phi_1 \in V_1$ such that $[\Sigma, \nu_1^{e_1} \circ f_1, \Sigma_1^{e_1}, \nu_1^{e_1} \circ \phi_1] \in F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$. Then, there exists a $\Delta \subset \mathcal{S}_1^{-1}(\mathcal{S}_1(\Delta_1) \cap \mathcal{S}_2(\Delta_2))$ containing e_1, and an open set $E_1' \subseteq \zeta_1(E_1)$ containing $\overline{\zeta_1 \circ \phi_1(\mathbb{D})}$, such that $\mathcal{H}$ is holomorphic in ϵ and z on $\Delta \times E_1'$ and $\mathcal{G}(\epsilon, z) = (\alpha(\epsilon), \mathcal{H}(\epsilon, z))$ is a biholomorphism onto $\mathcal{G}(\Delta \times E_1')$.

Proof. Let N be the connected component of $\mathcal{S}_1(\Delta_1) \cap \mathcal{S}_2(\Delta_2)$ that contains $\mathcal{S}_1(e_1)$. For $i = 1, 2$, let $\Delta_i' = \mathcal{S}_i^{-1}(N)$,

$$E_i^{\epsilon_i} = \nu_i^{\epsilon_i}(E_i)$$

and

$$A_i^{\epsilon_i} = (\nu_i^{\epsilon_i} \circ \phi_1)(\mathbb{D}).$$

Note that $\overline{A_i^{\epsilon_i}} \subset E_i^{\epsilon_i}$. By construction Δ_1' contains e_1.

Let

$$\widetilde{E}_i = \{(\epsilon_i, z) : \epsilon_i \in \Delta_i', z \in E_i^{\epsilon_i}\}$$

and

$$\widetilde{A}_i = \{(\epsilon_i, z) : \epsilon_i \in \Delta_i', z \in A_i^{\epsilon_i}\}.$$

Both of these sets are open by definition of $S(\Omega_i, D_i)$.

Now by Lemma 3.16 there is a biholomorphism $\beta : S(\Delta_1', D_1) \to S(\Delta_2', D_2)$ and moreover, $\beta(\widetilde{A}_1) = \widetilde{A}_2$. The last assertion follows from the definition of equivalence in the rigged Teichmüller space.

Let

$$\widetilde{C} = \beta^{-1}(\widetilde{E}_2) \cap \widetilde{E}_1$$

and note that $\overline{\widetilde{A}_1} \subset \widetilde{C}$.

Since $\widetilde{C}$ is open, so is

$$J = \tilde{\zeta}_1(\widetilde{C}) \subset \Delta_1' \times \zeta_1(E_1),$$

where $\tilde{\zeta}_1$ is defined in (3.6). Let $J^\epsilon = \{z : (\epsilon, z) \in J\}$. Then

$$\overline{\psi_1(\mathbb{D})} \subset J^\epsilon \subset \zeta_1(E_1)$$

for all ϵ, where $\psi_1 = \zeta_1 \circ \phi_1$. By the definition of $\widetilde{C}$, $\mathcal{H}$ is defined on J^ϵ.

We claim that there are connected open sets Δ and E' such that the closure of $\Delta \times E'$ is contained in J, $e_1 \in \Delta$ and $\overline{\psi_1(\mathbb{D})} \subset E'$. Since J is open and $\{e_1\} \times \overline{\psi_1(\mathbb{D})}$ is compact the existence of such sets Δ and E' follow from a standard topological argument.

Since $\mathcal{H}$, and therefore $\mathcal{G}$ are defined on J they are defined on $\Delta \times E'$. We will prove that $\mathcal{G}$ is biholomorphic by showing that it is equal to β expressed in terms of local coordinates. Using the coordinates defined in (3.6), noting that on E', $\nu^\epsilon = \iota^\epsilon$, and applying Lemma 3.16, we have for $(\epsilon, z) \in \Delta \times E'$ that

$$\begin{aligned}(\tilde{\zeta}_2 \circ \beta \circ \tilde{\zeta}_1^{-1})(\epsilon, z) &= \Big(\alpha(\epsilon), (\zeta_2 \circ (\nu_2^{\alpha(\epsilon)})^{-1} \circ \sigma_\epsilon \circ \nu_1^\epsilon \circ \zeta_1^{-1})(z)\Big) \\ &= (\alpha(\epsilon), \mathcal{H}(\epsilon, z)) \\ &= \mathcal{G}(\epsilon, z).\end{aligned}$$

Since β is a biholomorphism we see that on the domain $\Delta \times E'$, $\mathcal{G}$ is a biholomorphism and $\mathcal{H}$ is holomorphic. □

Theorem 3.26. *With notation as in Lemma 3.25, assume that* $p = [\Sigma, \nu_1^{e_1} \circ f_1, \Sigma_1^{e_1}, \nu_1^{e_1} \circ \phi_1]$ *is an arbitrary point in* $F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$*. There exists a* $V_1' \in \mathcal{V}_1$ *and a* Δ_1' *such that*

(1) $p \in F(V_1', S_1, \Delta_1') \subseteq F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$
(2) *For all* $\psi \in U_1'$ *(where* U_1' *is associated to* V_1' *as in Definition 2.28),* $\overline{\psi(\mathbb{D})}$ *is contained in an open set* E' *satisfying the consequences of Lemma 3.25*
(3) $G_2^{-1} \circ G_1$ *is holomorphic on* $\Delta_1' \times U_1'$*.*

Proof. By Lemma 3.25, there is an open set $\Delta_1'' \times E_1'$ such that $\overline{\zeta_1 \circ \phi_1(\mathbb{D})} \subset E_1'$, $e_1 \in \Delta_1''$, $\mathcal{H}$ is holomorphic on $\Delta_1'' \times E_1'$ and $\mathcal{G}$ is biholomorphic on $\Delta_1'' \times E_1'$. This immediately implies that there is an open set $\Delta_2' \times E_2' \subset \mathcal{G}(\Delta_1'' \times E_1')$ such that $\alpha(e_1) \in \Delta_2'$ and for $\psi_2 = H(e_1, \zeta_1 \circ \phi_1)$, $\overline{\psi_2(\mathbb{D})} \subseteq E_2'$. Now let $W_2 = \{\psi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{\psi(\mathbb{D})} \subseteq E_2'\}$. By Theorem 2.24 and Remark 2.35, $W_2 \cap U_2$ is open in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Note that $H(e_1, \zeta_1 \circ \phi_1) \in W_2 \cap U_2$.

Choose a compact set $K \subset E_1'$ which contains $\overline{\zeta_1 \circ \phi_1(\mathbb{D})}$ in its interior K_{int}. If we let $W_1 = \{\psi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} : \overline{\psi(\mathbb{D})} \subseteq K_{int}\}$, then W_1 is open by Theorem 2.24. We claim that H is holomorphic on $\Delta_1' \times W_1$. By Hartogs' theorem (see [16] for a version in a suitably general setting), it is enough to check holomorphicity separately in ϵ and ψ. By Lemma 2.5, H is holomorphic in ϵ for fixed ψ. On the other hand, by Theorem 2.25, H is holomorphic in ψ for fixed ϵ by our careful choice of W_1.

In particular, H is continuous and therefore $H^{-1}(W_2 \cap U_2) \cap (\Delta_1'' \times (W_1 \cap U_1))$ is open and contains $(e_1, \zeta_1 \circ \phi_1)$, hence we may choose an open subset $\Delta_1' \times U_1'$ containing $(e_1, \zeta_1 \circ \phi_1)$. Let V_1' be the element of $\mathcal{V}_1$ associated to U_1'. Clearly $U_1' \subseteq U_1$, and $H(\Delta_1' \times U_1') \subseteq U_2$ by construction; thus $F(V_1', S_1, \Delta_1') \subseteq F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$ so the first condition is satisfied. By construction, (2) is also satisfied. Since $U_1' \subseteq W_1$, H is holomorphic on $\Delta_1' \times U_1'$ and the fact that α is holomorphic on Δ' yields that $G_2^{-1} \circ G_1$ is holomorphic on $\Delta_1' \times U_1'$. This concludes the proof. □

Theorem 3.27. *The set* $\mathcal{F}$ *is a base for a Hausdorff, separable topology on* $\widetilde{T}_{\mathrm{WP}}(\Sigma)$*. Furthermore, with the atlas of charts given by (3.7),* $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ *is a Hilbert manifold.*

Proof. It follows directly from part (1) of Theorem 3.26 that $\mathcal{F}$ is a base for a topology on $\tilde{T}_{\mathrm{WP}}(\Sigma)$. From part (3), we have that the inverses of the maps (3.7) form an atlas with holomorphic transition functions. Thus it remains only to show that this topology is Hausdorff and separable. We first show that it is Hausdorff.

For $i = 1, 2$, let $p_i = [\Sigma, \nu_i^{e_i} \circ f_i, \Sigma_i^{e_i}, \nu_i^{e_i} \circ \phi_i]$ be distinct points, in sets $F(V_i, S_i, \Delta_i)$. If $F(V_i, S_i, \Delta_i)$ are disjoint, we are done. If not, by Lemma 3.16 setting Δ_i' to be the connected component of $\mathcal{S}_i^{-1}(\mathcal{S}_1(\Delta_1)\cap\mathcal{S}_2(\Delta_2))$ containing e_i, there is a biholomorphism $\alpha:\Delta_1'\to\Delta_2'$ such that $[\Sigma,\nu_2^{\alpha(\epsilon)}\circ f_2,\Sigma_2^{\alpha(\epsilon)}]=[\Sigma,\nu_1^{\epsilon}\circ f_1,\Sigma_1^{\epsilon}]$ for all $\epsilon\in\Delta_1'$.

There are two cases to consider. If $[\Sigma,\nu_2^{\alpha(e_1)}\circ f_2,\Sigma_2^{\alpha(e_1)}]\neq[\Sigma,\nu_2^{e_2}\circ f_2,\Sigma_2^{e_2}]$, then one can find $\Omega_1\subset\Delta_1$ and $\Omega_2\subset\Delta_2$ such that $\mathcal{S}_1(\Delta_1)$ and $\mathcal{S}_2(\Delta_2)$ are disjoint and $F(V_i,S_i,\Omega_i)$ still contains $[\Sigma,\nu_1^{e_i}\circ f_1,\Sigma_1^{e_i},\nu_1^{e_i}\circ\phi_i]$ for $i=1,2$. But then $F(V_i,S_i,\Omega_i)$ are disjoint, which takes care of the first case.

If on the other hand $[\Sigma,\nu_2^{\alpha(e_1)}\circ f_2,\Sigma_2^{\alpha(e_1)}]=[\Sigma,\nu_2^{e_2}\circ f_2,\Sigma_2^{e_2}]$, then by Theorem 3.26 there are sets $F(V_1',S_1,\Omega_1)$ and $F(V_2',S_1,\Omega_2)$ in $F(V_1,S_1,\Delta_1)\cap F(V_2,S_2,\Delta_2)$ containing p_1 and p_2 respectively. Thus we may write

$$p_1=[\Sigma,\nu_1^{e_1}\circ f_1,\Sigma_1^{e_1},\nu_1^{e_1}\circ\psi_1]\quad\text{and}\quad p_2=[\Sigma,\nu_1^{e_1}\circ f_1,\Sigma_1^{e_1},\nu_1^{e_1}\circ\psi_2].$$

For $i=1,2$, let U_i' be the subsets of $(\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})^n$ associated with V_i' as in Definition 2.28. Since $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ is an open subset of a Hilbert space, it is Hausdorff, so there are open sets W_i in U_i' containing p_i for $i=1,2$ and such that $W_1\cap W_2$ is empty. In that case if V_i'' are the elements of $\mathcal{V}$ associated to W_i, then $V_1''\cap V_2''$ is empty. This in turn implies that $F(V_1'',S_1,\Omega_1)\cap F(V_2'',S_1,\Omega_2)$ is empty which proves the claim in the second case.

We now prove that $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ is separable. Since $T(\Sigma)$ is a finite dimensional complex manifold it is, in particular, separable. Choose a countable dense subset $\mathfrak{A}$ of $T(\Sigma)$. For each $p=[\Sigma,f_1,\Sigma_1]\in\mathfrak{A}$, choose a specific representative (Σ,f_1,Σ_1). The space $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_2)$ is second countable and, in particular, it has a countable dense subset $\mathfrak{B}_p(\Sigma_1)$. Now if (Σ,f_2,Σ_2) is any other representative, there exists a unique biholomorphism $\sigma:\Sigma_1\to\Sigma_2$ (if σ_1 is another such biholomorphism, since by hypothesis $\sigma_1^{-1}\circ\sigma$ is homotopic to the identity and $2g-2+n>0$, it follows from Theorem 3.3 that $\sigma_1^{-1}\circ\sigma$ is the identity). We set

$$\mathfrak{B}_p(\Sigma_2)=\{(\sigma\circ\phi_1,\dots,\sigma\circ\phi_n)\,:\,(\phi_1,\dots,\phi_n)\in\mathfrak{B}_p(\Sigma_1)\}.$$

This is easily seen to be itself a countable dense set in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_2)$ and it is not hard to see that

$$\Upsilon=\{[\Sigma,f_1,\Sigma_1,\psi_1]\,:\,[\Sigma,f_1,\Sigma_1]\in\mathfrak{A},\ \psi_1\in\mathfrak{B}_p(\Sigma_1)\}$$

is well-defined. We will show that it is dense. Note that for any fixed $[\Sigma,f_1,\Sigma_1]$, the set of $[\Sigma,f_1,\Sigma_1,\psi_1]\in\Upsilon$ is entirely determined by any particular representative (Σ,f_1,Σ_1), and so this is a countable set.

Let $F(V,S,\Delta)\in\mathcal{F}$. Since $\mathfrak{A}$ is dense, there is some $[\Sigma,f_2,\Sigma_2]\in\mathfrak{A}\cap S(\Delta)$. For a specific representative (Σ,f_2,Σ_2) there is a $\psi_2\in\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_2)$ such that $[\Sigma,f_2,\Sigma_2,\psi_2]\in F(V,S,\Delta)$. By Lemma 3.21 the set of points in F over $[\Sigma,f_2,\Sigma_2]$ is open. Thus since $\mathfrak{B}_p(\Sigma_2)$ is dense in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_2)$ there is a $\psi_3\in\mathfrak{B}_p(\Sigma_2)$ such that $[\Sigma,f_2,\Sigma_2,\psi_3]\in F$. By definition $[\Sigma,f_2,\Sigma_2,\psi_3]\in\Upsilon$, which completes the proof. □

Remark 3.28. It can be shown that $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ is second countable. The proof involves somewhat tedious notational difficulties, so we only give a sketch of the proof. No results in this paper depend on second countability of $\widetilde{T}_{\mathrm{WP}}(\Sigma)$.

Fix a countable basis $\mathfrak{O}$ for $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. For any $[\Sigma,f_1,\Sigma_1]\in\mathfrak{A}$, choose a representative (Σ,f_1,Σ_1), and fix the following objects. Let $\mathfrak{C}(\Sigma_1)$ be a countable collection of n-charts on Σ_1 constructed as in the proof of Theorem 2.29. Let $\mathcal{V}_c(\Sigma_1)$ be the countable dense subset of $\mathcal{V}(\Sigma_1)$ corresponding to $\mathfrak{O}$ and $\mathfrak{C}(\Sigma_1)$ as in the proof of Theorem 2.29. Finally, fix a countable base $\mathfrak{B}(\Sigma_1)$ of open sets in Σ_1.

Now if (Σ,f_2,Σ_2) is any other representative, there is a unique biholomorphism $\sigma:\Sigma_1\to\Sigma_2$ as in the proof of Theorem 3.27. Transfer each of the preceding objects to Σ_2

by composition with σ in the appropriate way; for example, $\mathfrak{C}(\Sigma_2)$ is the set of n-charts $(\zeta_1 \circ \sigma^{-1}, \sigma(E_1), \ldots, \zeta_n \circ \sigma^{-1}, \sigma(E_n))$ and so on. Finally fix a countable base $\mathfrak{D}$ of $\mathbb{C}^n$ (for example, the set of discs of rational radius centered at rational points).

We now define the subset $\mathcal{F}_c$ of $\mathcal{F}$ to be the set of $F(V, S, \Delta) \in \mathcal{F}$ such that

(1) the variation $S(\Omega)$ is based at a point $[\Sigma, f_1, \Sigma_1] \in \mathfrak{A}$
(2) $S(\Omega)$ is compatible with some fixed n-chart in $\mathfrak{C}(\Sigma_1)$
(3) Ω and Δ are both in $\mathfrak{D} \times \cdots \times \mathfrak{D}$
(4) $V \in \mathcal{V}(\Sigma_1)$.

The set $\mathcal{F}_c$ is countable by construction, and does not depend on the choice of representative. It can be shown with some work that $\mathcal{F}_c$ is a base compatible with $\mathcal{F}$.

3.6. **Compatibility with the non-WP rigged Teichmüller space.** In [21] the following rigged Teichmüller space was defined.

Definition 3.29. Let $\widetilde{T}(\Sigma)$ be defined by replacing $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1)$ with $\mathcal{O}^{\mathrm{qc}}(\Sigma_1)$ in Definition 3.2.

It was shown in [22] that $\widetilde{T}(\Sigma)$ is a complex Banach manifold with charts as in Definition 3.22 with $U \subset (\mathcal{O}^{\mathrm{qc}})^n$, and $\mathcal{O}^{\mathrm{qc}}$ replacing $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ in all the preceding definitions and constructions. Furthermore, the complex structure on $\mathcal{O}^{\mathrm{qc}}$ is given by the embedding χ defined by (2.2). We use the same notation for the charts and constructions on $\widetilde{T}(\Sigma)$ as for $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ without further comment.

The complex structures on $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ and $\widetilde{T}(\Sigma)$ are compatible in the following sense.

Theorem 3.30. *The inclusion map $I_T : \widetilde{T}_{\mathrm{WP}}(\Sigma) \to \widetilde{T}(\Sigma)$ is holomorphic.*

Proof. Choose any point $[\Sigma, f, \Sigma_*, \phi] \in \widetilde{T}_{\mathrm{WP}}(\Sigma)$. There is a parametrization $G : \Omega \times U \to \widetilde{T}(\Sigma)$ onto a neighborhood of this point (see Definition 3.22). We choose U small enough that ν^ϵ is holomorphic on $\overline{\phi(\mathbb{D})}$ for all $\phi \in U$.

Let $W = \chi^n(U)$ where $\chi^n : \mathcal{O}^{\mathrm{qc}} \times \cdots \times \mathcal{O}^{\mathrm{qc}} \to \bigoplus^n (A_1^\infty(\mathbb{D}) \oplus \mathbb{C})$ is defined by

$$\chi^n(\phi_1, \ldots, \phi_n) = (\chi(\phi_1), \ldots, \chi(\phi_n)).$$

Define $F : \Omega \times W \to \widetilde{T}(\Sigma)$ by

$$F = G \circ (\mathrm{id}, (\chi^n)^{-1})$$

where id is the identity map on Ω. These are coordinates on $\widetilde{T}(\Sigma)$.

Let $W_{\mathrm{WP}} = W \cap \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} = \iota^{-1}(W)$ (recall that ι is the inclusion map of $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ in $\mathcal{O}^{\mathrm{qc}}$). The set W_{WP} is open by Theorem 2.3. We further have that $F(\Omega \times W_{\mathrm{WP}}) = \widetilde{T}_{\mathrm{WP}} \cap W$. To see this note that $F(\Omega \times W_{\mathrm{WP}}) = G(\Omega \times (\chi^n)^{-1}(W_{\mathrm{WP}}))$. By definition $\nu^\epsilon \circ \phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$ if and only if for a parameter $\eta : A \to \mathbb{C}$ defined on an open neighborhood A of $\overline{\nu^\epsilon(\phi(\mathbb{D}))}$ it holds that $\eta \circ \nu^\epsilon \circ \phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. This holds if and only if $\phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$ since ν^ϵ is holomorphic on a neighborhood of $\overline{\phi(\mathbb{D})}$.

It follows from Theorem 2.24 that $F^{-1} \circ I_T \circ F$ is holomorphic. Since F are local coordinates, I_T is holomorphic on the image of F. Since coordinates of the form F cover $\widetilde{T}(\Sigma)$, this proves the theorem. $\square$

Note that this does not imply that $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ is a complex submanifold of $\widetilde{T}(\Sigma)$.

4. A WP-class Teichmüller space of bordered surfaces

We are at last in a position to define the WP-class Teichmüller space of a bordered surface and demonstrate that it has a natural complex Hilbert manifold structure. In Section 4.1 we define the WP-class Teichmüller space $T_{\mathrm{WP}}(\Sigma^B)$ of a bordered surface Σ^B, and define some "modular groups" which act on it. In Section 4.2 we show how to obtain a punctured surface by sewing "caps" onto the bordered surface using the riggings. It is also demonstrated that sewing on caps takes the WP-class Teichmüller space into the WP-class rigged Teichmüller space $\widetilde{T}_{\mathrm{WP}}(\Sigma)$. In Section 4.3 we prove that the WP-class Teichmüller space of bordered surfaces is a Hilbert manifold. We do this by showing that the WP-class rigged Teichmüller space $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ is a quotient of $T_{\mathrm{WP}}(\Sigma^B)$ by a properly discontinuous, fixed point free group of local homeomorphisms, and passing the charts on $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ upwards. Finally, in Section 4.4 we show that the rigged moduli space of Friedan and Shenker is a Hilbert manifold. This follows from the fact that the rigged moduli space is a quotient of $T_{\mathrm{WP}}(\Sigma^B)$ by a properly discontinuous fixed-point free group of biholomorphisms.

4.1. Definition of the WP-class Teichmüller space and modular groups.

The reader is referred to Section 2.2 for some of the notation and definitions used below.

We now define the WP-class Teichmüller space of a bordered Riemann surface which is obtained by replacing the quasiconformal marking maps in the usual Teichmüller space (see Definition 3.1) with WP-class quasiconformal maps.

Definition 4.1. Fix a bordered Riemann surface Σ^B of type (g, n). Let

$$T_{\mathrm{WP}}(\Sigma^B) = \{(\Sigma^B, f, \Sigma_1^B)\}/\sim$$

where Σ_1^B is a bordered Riemann surface of the same type, $f \in \mathrm{QC}_0(\Sigma^B, \Sigma_1^B)$, and two triples $(\Sigma^B, f_i, \Sigma_i^B)$, $i = 1, 2$ are equivalent if there is a biholomorphism $\sigma : \Sigma_1^B \to \Sigma_2^B$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary.

The space $T_{\mathrm{WP}}(\Sigma^B)$ is called the WP-*class Teichmüller space* and its elements are denoted by equivalence classes of the form $[\Sigma^B, f_1, \Sigma_1^B]$.

An important ingredient in the construction of the complex Hilbert manifold structure is a kind of modular group (or mapping class group). To distinguish between the different possible boundary condition we use some slightly non-standard notation following [21]; we recall the definitions here.

Let Σ^B be a bordered Riemann surface and $\mathrm{QCI}(\Sigma^B)$ denote the set of quasiconformal maps from Σ^B onto Σ^B which are the identity on the boundary. This is a group which acts on the marking maps by right composition. Let $\mathrm{QCI}_n(\Sigma^B)$ denote the subset of $\mathrm{QCI}(\Sigma^B)$ which are homotopic to the identity rel boundary (the subscript n stands for "null-homotopic").

Definition 4.2. Let $\mathrm{PModI}(\Sigma^B) = \mathrm{QCI}(\Sigma^B)/\sim$ where two elements f and g of $\mathrm{QCI}(\Sigma^B)$ are equivalent ($f \sim g$) if and only if $f \circ g^{-1} \in \mathrm{QCI}_n(\Sigma)$.

The "P" stands for "pure", which means that the mappings preserve the ordering of the boundary components, and "I" stands for "identity".

There is a natural action of $\mathrm{PModI}(\Sigma^B)$ on $T(\Sigma^B)$ by right composition, namely

$$[\rho][\Sigma^B, f, \Sigma_1^B] = [\Sigma^B, f \circ \rho, \Sigma_1^B]. \tag{4.1}$$

This is independent of the choice of representative $\rho \in \mathrm{QCI}(\Sigma^B)$ of $[\rho] \in \mathrm{PModI}(\Sigma^B)$. It is a standard fact that $\mathrm{PModI}(\Sigma^B)$ is finitely generated by Dehn twists. Using these twists we can define two natural subgroups of $\mathrm{PModI}(\Sigma^B)$ (see [21] for details).

Definition 4.3. Let Σ^B be a bordered Riemann surface. Let $\mathrm{DB}(\Sigma^B)$ be the subgroup of $\mathrm{PModI}(\Sigma^B)$ generated by Dehn twists around simple closed curves Σ which are homotopic to a boundary curve. Let $\mathrm{DI}(\Sigma^B)$ be the subgroup of $\mathrm{PModI}(\Sigma^B)$ generated by Dehn twists around simple closed curves in Σ^B which are neither homotopic to a boundary curve nor null-homotopic.

Here "B" stands for "boundary" and "I" stands for "internal".

The next Lemma implies that we can consider $\mathrm{PModI}(\Sigma^B)$ and $\mathrm{DB}(\Sigma^B)$ as acting on $T_{\mathrm{WP}}(\Sigma^B)$.

Lemma 4.4. *Every element of* $\mathrm{QCI}(\Sigma^B)$ *is in* $\mathrm{QC}_0(\Sigma^B, \Sigma^B)$. *Thus, the group action of* $\mathrm{PModI}(\Sigma^B)$ *on* $T(\Sigma^B)$ *preserves* $T_{\mathrm{WP}}(\Sigma^B)$.

Proof. The first statement follows from Definition 2.18, and Definition 2.14 with $H_1 = H_2$. The second statement follows from Proposition 2.19. □

4.2. **Sewing on caps.** Given a bordered Riemann surface Σ^B together with quasisymmetric parametrizations of its boundaries by the circle, one can sew on copies of the punctured disk to obtain a punctured Riemann surface Σ. The collection of parametrizations extend to an element of $\mathcal{O}^{\mathrm{qc}}(\Sigma)$. In [21], two of the authors showed that this operation can be used to exhibit a natural correspondence between the rigged Teichmüller space $\widetilde{T}(\Sigma)$ and the Teichmüller space $T(\Sigma^B)$, and showed in [23] that this results in a natural fiber structure on $T(\Sigma^B)$. We will be using this fiber structure as the principle framework for constructing the Hilbert manifold structure on $T_{\mathrm{WP}}(\Sigma^B)$. It is thus necessary to describe sewing on caps here, in the setting of WP-class quasisymmetries.

Definition 4.5. Let Σ^B be a bordered Riemann surface with boundary curves C_i, $i = 1, \ldots, n$. The *riggings* of Σ^B is the collection $\mathrm{Rig}(\Sigma^B)$ of n-tuples $\psi = (\psi_1, \ldots, \psi_n)$ such that $\psi_i \in \mathrm{QS}(\mathbb{S}^1, C_i)$. The WP-class riggings is the collection $\mathrm{Rig}_{\mathrm{WP}}(\Sigma^B)$ of n-tuples $\psi = (\psi_1, \ldots, \psi_n)$ such that $\psi_i \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, C_i)$

Let Σ^B be a fixed bordered Riemann surface of type (g, n) say, and $\psi \in \mathrm{Rig}(\Sigma^B)$. Let $\mathbb{D}_0$ denote the punctured unit disk $\mathbb{D}\backslash\{0\}$. We obtain a new topological space

$$\Sigma = \overline{\Sigma^B} \sqcup \overline{\mathbb{D}}_0 \sqcup \cdots \sqcup \overline{\mathbb{D}}_0 / \sim . \tag{4.2}$$

Here we treat the n copies of $\mathbb{D}_0$ as distinct and ordered, and two points p and q are equivalent($p \sim q$) if p is in the boundary of the ith disk, q is in the ith boundary C_i, and $q = \psi_i(p)$. By [21, Theorems 3.2, 3.3] this topological space has a unique complex structure which is compatible with the complex structures on Σ^B and each copy of $\mathbb{D}_0$. We will call the image of a boundary curve in Σ under inclusion (which is also the image of $\partial\mathbb{D}$ under inclusion) a *seam*. We will call the copy of each disk in Σ a *cap*. Finally, we will denote equation (4.2) by

$$\Sigma = \Sigma^B \#_\psi \mathbb{D}_0^n$$

to emphasize the underlying element of $\mathrm{Rig}(\Sigma^B)$ used to sew.

For each $i = 1, \ldots, n$ the map ψ_i can be extended to a map $\tilde{\psi}_i : \overline{\mathbb{D}}_0 \to \Sigma$ defined by

$$\tilde{\psi}_i(z) = \begin{cases} \psi(z), & \text{for } z \in \partial\mathbb{D} \\ z, & \text{for } z \in \mathbb{D}. \end{cases} \tag{4.3}$$

Note that $\tilde{\psi}_i$ is well defined and continuous because the map ψ_i is used to identify $\partial\mathbb{D}$ with C_i. Moreover, $\tilde{\psi}$ is holomorphic on $\mathbb{D}_0$. It is important to keep in mind that if the seam in Σ is viewed as $\partial\mathbb{D}$ then in fact $\tilde{\psi}_i$ is also the identity on $\partial\mathbb{D}$.

Remark 4.6. The complex structure on the sewn surface is easily described in terms of conformal welding. Choose a seam C_i and let H be a collar chart (see Definition 2.13) with respect to C_i with domain A say. We have that $H \circ \psi_i$ is in $\mathrm{QS}(\mathbb{S}^1)$. Let $F : \mathbb{D} \to \mathbb{C}$ and $G : \mathbb{D}^* \to \overline{\mathbb{C}}$ be the unique holomorphic welding maps such that $G^{-1} \circ F = H \circ \psi_i$ when restricted to $\mathbb{S}^1$, $F(0) = 0$, $G(\infty) = \infty$ and $G'(\infty) = 1$. Note that F and G have quasiconformal extensions to $\mathbb{C}$ and $\overline{\mathbb{C}}$ respectively.

Let ζ_i be the continuous map on $A \cup \overline{\tilde{\psi}_i(\mathbb{D})}$ defined by

$$\zeta_i = \begin{cases} F \circ \tilde{\psi}_i^{-1} & \text{on } \tilde{\psi}(\mathbb{D}) \\ G \circ H & \text{on } A. \end{cases} \tag{4.4}$$

It is easily checked that there is such a continuous extension. Since ζ_i is 0-quasiconformal on $\tilde{\psi}_i(\mathbb{D})$ and A, by removability of quasicircles [15, V.3] ζ_i is 0-quasiconformal (that is, holomorphic and one-to-one), on $A \cup \overline{\tilde{\psi}_i(\mathbb{D})}$. Thus ζ is a local coordinate on Σ containing the closure of the cap.

The crucial fact about the extension $\tilde{\psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_n)$ is that it is in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$. In fact we have the following proposition.

Proposition 4.7. *Let Σ^B be a bordered Riemann surface, and $\psi = (\psi_1, \ldots, \psi_n)$ be an element of $\mathrm{QS}(\mathbb{S}^1, \Sigma^B)$. Let $\Sigma = \Sigma^B \#_\psi \mathbb{D}_0^n$ and $\tilde{\psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_n)$ be the n-tuple of holomorphic extensions to $\mathbb{D}_0$. Then $\psi \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma^B)$ if and only if $\tilde{\psi} \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)$.*

Proof. Let H be a collar chart with respect to the ith boundary curve C_i, and let F, G and ζ_i be as in Remark 4.6. By definition $\psi_i \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, C_i)$ if and only if $H \circ \psi_i \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ which holds if and only if the welding map F is in $\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$. Since $F = \zeta \circ \tilde{\psi}_i$ this proves the claim. □

The following Proposition is a consequence of Proposition 2.20 and Theorem 2.10.

Proposition 4.8. *Let Σ_1^B and Σ_2^B be bordered Riemann surfaces, and let $\tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma_1^B)$. Then $f \in \mathrm{QC}_0(\Sigma_1^B, \Sigma_2^B)$ if and only if $f \circ \tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma_2^B)$.*

We now have enough tools to describe the relation between $T_{\mathrm{WP}}(\Sigma^B)$ and $\widetilde{T}_{\mathrm{WP}}(\Sigma)$.

Definition 4.9. Let Σ^B be a bordered Riemann surface, let $\tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma)$ be a fixed rigging, and let $\Sigma = \Sigma^B \#_\tau \mathbb{D}_0^n$. We define

$$\begin{aligned} \Pi : T(\Sigma^B) &\longrightarrow \widetilde{T}(\Sigma) \\ [\Sigma^B, f, \Sigma_1^B] &\longmapsto [\Sigma, \tilde{f}, \Sigma_1, \tilde{f} \circ \tilde{\tau}]. \end{aligned}$$

where $\tilde{\tau}$ is the extension defined by (4.3),

$$\tilde{f}(z) = \begin{cases} f(z), & z \in \overline{\Sigma^B} \\ z, & z \in \text{ cap}, \end{cases} \tag{4.5}$$

and $\Sigma_1 = \Sigma_1^B \#_{f \circ \tau} \mathbb{D}_0^n$ is the Riemann surface obtained by sewing caps onto Σ_1^B using $f \circ \tau$.

The map $\tilde{f}$ is quasiconformal, since it is quasiconformal on Σ^B and the cap, and is continuous on the seam [15, V.3].

Remark 4.10. If $\widetilde{f \circ \tau}$ denotes the holomorphic extension of $f \circ \tau$ as in equation (4.3), then $\widetilde{f \circ \tau} = \tilde{f} \circ \tilde{\tau}$.

It was shown in [21] that Π is invariant under the action of DB, and in fact

$$\Pi([\Sigma^B, f, \Sigma_1^B]) = \Pi([\Sigma^B, f_2, \Sigma_2^B]) \iff [\Sigma^B, f_2, \Sigma_2^B] = [\rho][\Sigma^B, f_1, \Sigma_1^B]$$

for some $[\rho] \in \mathrm{DB}$. (The reader is warned that the direction of the riggings in [21] is opposite to the convention used here). Thus $\widetilde{T}(\Sigma) = T(\Sigma^B)/\,\mathrm{DB}$ as sets. Furthermore, the group action by DB is properly discontinuous and fixed point free, and the map Π is holomorphic with local holomorphic inverses. Thus $\widetilde{T}(\Sigma)$ inherits a complex structure from $T(\Sigma^B)$.

On the other hand, in the WP-class setting, instead of having a complex structure on Teichmüller space in the first place, we are trying to construct one. In the next section, we will reverse the argument above and lift the complex Hilbert manifold structure on $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ to $T_{\mathrm{WP}}(\Sigma^B)$. To this end we need the following facts.

Proposition 4.11. *Let $p = [\Sigma^B, f, \Sigma_1^B] \in T(\Sigma^B)$. Then $p \in T_{\mathrm{WP}}(\Sigma^B)$ if and only if $\Pi(p) \in \widetilde{T}_{\mathrm{WP}}(\Sigma)$.*

Proof. Since $\tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma^B)$, $f \in \mathrm{QC}_0(\Sigma^B, \Sigma_1^B)$ if and only if $f \circ \tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma_1^B)$ by Proposition 4.8. And this holds if and only if $\widetilde{f \circ \tau} \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1)$ by Proposition 4.7. By Remark 4.10, $\tilde{f} \circ \tilde{\tau} \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma_1)$ which proves the claim. □

We now define the map Π_{WP} by

$$\Pi_{\mathrm{WP}} = \Pi|_{T_{\mathrm{WP}}(\Sigma^B)},$$

and as a result of this proposition we have

$$\Pi_{\mathrm{WP}} : T_{\mathrm{WP}}(\Sigma^B) \longrightarrow \widetilde{T}_{\mathrm{WP}}(\Sigma). \tag{4.6}$$

Proposition 4.12. *The action of* DB *is fixed point free, and for $[\Sigma^B, f_i, \Sigma_i^B] \in T_{\mathrm{WP}}(\Sigma^B)$, $i = 1, 2$, $\Pi_{\mathrm{WP}}([\Sigma^B, f_1, \Sigma_1^B]) = \Pi_{\mathrm{WP}}([\Sigma^B, f_2, \Sigma_2^B])$ if and only if there is a $[\rho] \in \mathrm{DB}$ such that $[\rho][\Sigma^B, f_1, \Sigma_1^B] = [\Sigma^B, f_2, \Sigma_2^B]$. The map $\Pi : T_{\mathrm{WP}}(\Sigma^B) \to \widetilde{T}_{\mathrm{WP}}(\Sigma)$ is onto and thus, as sets, $T_{\mathrm{WP}}(\Sigma^B)/\,\mathrm{DB}$ and $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ are in one-to-one correspondence.*

Proof. These claims are all true in the non-WP setting [21, Lemma 5.1, Theorem 5.6]. Thus by Proposition 4.11 they are true in the WP-class setting. □

4.3. **Complex Hilbert manifold structure on WP-class Teichmüller space.** Next we describe how to construct the complex structure on $T_{\mathrm{WP}}(\Sigma^B)$. Let Σ^B be a bordered Riemann surface, and let $\tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma^B)$. Let Σ be the Riemann surface obtained by sewing on caps via τ as in the previous section.

We define a base $\mathcal{B}$ for a topology on $T_{\mathrm{WP}}(\Sigma^B)$ as follows. Recall that $\mathcal{F}$ is the base for $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ (Definition 3.20).

Definition 4.13. A set $B \in \mathcal{B}$ if and only if

(1) $\Pi_{\mathrm{WP}}(B) \in \mathcal{F}$
(2) Π_{WP} is one-to-one on B.

Theorem 4.14. *The set $\mathcal{B}$ is a base. With the topology corresponding to $\mathcal{B}$, $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ has the quotient topology with respect to Π_{WP} and DB is properly discontinuous.*

Proof. Let $x \in T_{\mathrm{WP}}(\Sigma^B)$. We show that there is a $B \in \mathcal{B}$ containing x. There is a neighborhood U of x in $T(\Sigma^B)$ on which Π is one-to-one [21]. Let $U' = \Pi(U)$; this is open in $\widetilde{T}(\Sigma)$ [21]. By Theorem 3.30, the set $U' \cap \widetilde{T}_{\mathrm{WP}}(\Sigma)$ is open in $\widetilde{T}_{\mathrm{WP}}(\Sigma)$. Thus there is an element $F \subset U' \cap \widetilde{T}_{\mathrm{WP}}(\Sigma)$ of the base $\mathcal{F}$ which contains $\Pi(x)$. Since $\Pi|_U$ is invertible, we can set $B = (\Pi|_U)^{-1}(F)$, and B is in $\mathcal{B}$ and contains x.

Next, fix $q \in T_{\mathrm{WP}}(\Sigma^B)$ and let $B_1, B_2 \in \mathcal{B}$ contain q. We show that the intersection contains an element of $\mathcal{B}$. Let $U \subset \Pi_{\mathrm{WP}}(B_1) \cap \Pi_{\mathrm{WP}}(B_2)$ be a set in $\mathcal{F}$ containing $\Pi_{\mathrm{WP}}(q)$. Set $B_3 = \left(\Pi_{\mathrm{WP}}|_{B_1}\right)^{-1}(U) \subset B_1 \cap B_2$. We then have that Π_{WP} is one-to-one on B_3 (since $B_3 \subset B_1$) and $\Pi_{\mathrm{WP}}(B_3) = U$. So $B_3 \in \mathcal{B}$. Thus $\mathcal{B}$ is a base.

Now we show that $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ has the quotient topology with respect to Π_{WP}. Let U be open in $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ and let $x \in {\Pi_{\mathrm{WP}}}^{-1}(U)$. There is a $B_x \in \mathcal{B}$ containing x such that Π_{WP} is one-to-one on B_x, and $\Pi_{\mathrm{WP}}(B_x)$ is open and in $\mathcal{F}$. Since $\Pi_{\mathrm{WP}}(B_x) \cap U$ is open and non-empty (it contains $\Pi_{\mathrm{WP}}(x)$), there is a $F_x \in \mathcal{F}$ such that $\Pi_{\mathrm{WP}}(x) \in F_x$ and $F_x \subset \Pi_{\mathrm{WP}}(B_x) \cap U$. By definition $\tilde{B}_x = \left(\Pi_{\mathrm{WP}}|_{B_x}\right)^{-1}(F_x) \in \mathcal{B}$. By construction $x \in \tilde{B}_x$ and $\tilde{B}_x$ is open and contained in U. Since x was arbitrary, ${\Pi_{\mathrm{WP}}}^{-1}(U)$ is open.

Let $U \in \widetilde{T}_{\mathrm{WP}}(\Sigma)$ be such that ${\Pi_{\mathrm{WP}}}^{-1}(U)$ is open. Let $x \in U$ and $y \in {\Pi_{\mathrm{WP}}}^{-1}(U)$ be such that $\Pi_{\mathrm{WP}}(y) = x$. There is a $B_y \in \mathcal{B}$ such that $y \in B_y \subset {\Pi_{\mathrm{WP}}}^{-1}(U)$. So $\Pi_{\mathrm{WP}}(B_y) \subset U$ and $x \in \Pi_{\mathrm{WP}}(B_y)$. Since B_y is in $\mathcal{B}$, $\Pi_{\mathrm{WP}}(B_y) \in \mathcal{F}$, so $\Pi_{\mathrm{WP}}(B_y)$ is open. Since x was arbitrary, U is open. This completes the proof that $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ has the quotient topology.

Finally, we show that DB acts properly discontinuously on $T_{\mathrm{WP}}(\Sigma^B)$. Let $x \in T_{\mathrm{WP}}(\Sigma^B)$. By [21, Lemma 5.2], DB acts properly discontinuously on $T(\Sigma^B)$ in its topology. Thus there is an open set $U \subset T(\Sigma^B)$ containing x such that $g(U) \cap U$ is empty for all $g \in \mathrm{DB}$, and on which Π is one-to-one. Furthermore, $\Pi(U)$ is open in $\widetilde{T}(\Sigma)$ since Π is a local homeomorphism [21]. By Theorem 3.30, $\Pi(U) \cap \widetilde{T}_{\mathrm{WP}}(\Sigma)$ is open in $\widetilde{T}_{\mathrm{WP}}(\Sigma)$, so there exists an $F \in \mathcal{F}$ such that $F \subset \Pi(U) \cap \widetilde{T}_{\mathrm{WP}}(\Sigma)$ and $\Pi(x) \in F$ (note that $\Pi(x) \in \widetilde{T}_{\mathrm{WP}}(\Sigma)$ by Proposition 4.11). So $W = (\Pi|_U)^{-1}(F)$ is in $\mathcal{B}$ by definition, and contains x. In particular W is open, and since $W \subset U$ by construction, $g(W) \cap W$ is empty for all $g \in \mathrm{DB}$. This completes the proof. □

Corollary 4.15. *With the topology defined by $\mathcal{B}$, $T_{\mathrm{WP}}(\Sigma^B)$ is Hausdorff and separable.*

Proof. Let $x, y \in T_{\mathrm{WP}}(\Sigma^B)$, $x \neq y$. If $\Pi_{\mathrm{WP}}(x) \neq \Pi_{\mathrm{WP}}(y)$, then since $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ is Hausdorff by Theorem 3.27, there are disjoint open sets $F_x, F_y \in \mathcal{F}$ such that $\Pi_{\mathrm{WP}}(x) \in F_x$ and

$\Pi_{\mathrm{WP}}(y) \in F_y$. Since $\mathcal{B}$ is a base there are sets $B_x, B_y \in \mathcal{B}$ such that $x \in B_x$, $y \in B_y$, $\Pi_{\mathrm{WP}}(B_x) \subset F_x$ and $\Pi_{\mathrm{WP}}(B_y) \subset F_y$. Thus B_x and B_y are disjoint.

Now assume that $\Pi_{\mathrm{WP}}(x) = \Pi_{\mathrm{WP}}(y)$. Thus there is a non-trivial $[\rho] \in \mathrm{DB}$ such that $[\rho]x = y$. Since by Theorem 4.14 DB acts properly discontinuously there is an open set V containing x such that $[\rho]V \cap V$ is empty; $[\rho]V$ is open and contains y. This completes the proof that $T_{\mathrm{WP}}(\Sigma^B)$ is Hausdorff.

To see that $T_{\mathrm{WP}}(\Sigma^B)$ is separable, let $\mathfrak{A}$ be a countable dense subset of $\widetilde{T}_{\mathrm{WP}}(\Sigma)$. Define $\mathfrak{B} = \{p \in T_{\mathrm{WP}}(\Sigma^B) : \Pi(p) \in \mathfrak{A}\}$. Since DB is countable, $\mathfrak{B}$ is countable. To see that $\mathfrak{B}$ is dense, observe that if U is open in $T_{\mathrm{WP}}(\Sigma^B)$ then, since DB acts properly discontinuously by Theorem 4.14, there is a $V \subseteq U$ on which Π is a homeomorphism onto its image. So there is a $q \in \mathfrak{A} \cap \Pi(V)$, and thus for a local inverse Π^{-1} on $\Pi(V)$ we can set $p = \Pi^{-1}(q) \in V \cap \mathfrak{B} \subseteq U \cap \mathfrak{B}$. This completes the proof. □

Remark 4.16. It can also be shown that $T_{\mathrm{WP}}(\Sigma^B)$ is second countable. To see this, let $\mathcal{F}'$ be a countable base for $\widetilde{T}_{\mathrm{WP}}(\Sigma)$. Such a base exists by Remark 3.28. Let $\mathcal{B}' = \{B \in \mathcal{B} : \Pi_{\mathrm{WP}}(B) \in \mathcal{F}'\}$. It is elementary to verify that $\mathcal{B}'$ is a base. The fact that $\mathcal{B}'$ is countable follows from the facts that $\mathcal{F}'$ is countable and DB is countable. Indeed, for each element F of $\mathcal{F}'$ we can choose an element B_F of $\mathcal{B}'$. Each B in $\mathcal{B}'$ is $[\rho]B_F$ for some $F \in \mathcal{F}'$ and $\rho \in \mathrm{DB}$.

Using this base, we now define the charts on $T_{\mathrm{WP}}(\Sigma^B)$ that will give it a complex Hilbert space structure. For any $x \in T_{\mathrm{WP}}(\Sigma^B)$, let B be in the base $\mathcal{B}$; therefore $F = \Pi(B)$ is in the base $\mathcal{F}$ of $\widetilde{T}_{\mathrm{WP}}(\Sigma)$ (see Definition 3.20). From Definition 3.22 there is the chart $G^{-1} : F \to \mathbb{C}^d \otimes (\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})^n$, where $d = 3g-3+n$ is the dimension of $T(\Sigma)$ and n is the number of boundary curves of Σ^B.

Definition 4.17 (Charts for $T_{\mathrm{WP}}(\Sigma^B)$)**.** Given $x \in B \subset T_{\mathrm{WP}}(\Sigma^B)$ as above, we define the chart

$$S : B \longrightarrow \mathbb{C}^d \otimes (\mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}})^n$$

by $S = G^{-1} \circ \Pi_{\mathrm{WP}}$.

Note that to get a true chart into a Hilbert space we need to compose S with maps $\chi : \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}} \to A^2_1(\mathbb{D}) \oplus \mathbb{C}$ (see (2.2) and Theorem 2.3) as in the proof of Theorem 3.30.

Theorem 4.18. *The WP-class Teichmüller space $T_{\mathrm{WP}}(\Sigma^B)$ with charts given in the above definition is a complex Hilbert manifold. With this given complex structure, Π_{WP} is locally biholomorphic in the sense that for every point $x \in T_{\mathrm{WP}}(\Sigma^B)$ there is a neighborhood U of x such that Π_{WP} restricted to U is a biholomorphism onto its image.*

Proof. By Corollary 4.15, we need only to show that $T_{\mathrm{WP}}(\Sigma^B)$ is locally homeomorphic to a Hilbert space, and exhibit an atlas of charts with holomorphic transition functions. Since Definition 4.17 defines a chart for any $x \in T_{\mathrm{WP}}(\Sigma^B)$, the set of such charts clearly covers $T_{\mathrm{WP}}(\Sigma^B)$. The maps S are clearly homeomorphisms, since G's are biholomorphisms by Theorem 3.27 and Π_{WP}'s are local homeomorphisms by the definition of the topology on $\widetilde{T}_{\mathrm{WP}}(\Sigma)$.

Assume that two such charts (S, B) and (S', B') have overlapping domains. We show that $S' \circ S^{-1}$ is holomorphic on $B \cap B'$. Let $x \in B \cap B'$. Since $\mathcal{B}$ is a base, there is a $B_1 \in B \cap B'$ containing x. So Π is one-to-one on B_1; note also that the determination of Π^{-1} on $\Pi(B_1)$ agrees with those on $\Pi(B)$ and $\Pi(B')$. So $S' \circ S^{-1} = (G')^{-1} \circ \Pi \circ \Pi^{-1} \circ G = (G')^{-1} \circ G^{-1}$ which is holomorphic by Theorem 3.27. The same proof applies to $S \circ S'^{-1}$. □

The construction of the Hilbert manifold structure on $T_{\mathrm{WP}}(\Sigma^B)$ made use of an arbitrary choice of a *base rigging* $\tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma^B)$, but in fact the resulting complex structure is independent of this choice. We will show a slightly stronger result. If one considers a base Riemann surface together with a base rigging (Σ_b^B, τ_b) to define a base point, then the *change of base point* to another such pair (Σ_a^B, τ_a) is a biholomorphism. We proceed by first examining the change of base point map for $\widetilde{T}_{\mathrm{WP}}(\Sigma)$.

Fix two punctured Riemann surfaces Σ_a and Σ_b of the same topological type, and let $\alpha : \Sigma_a \to \Sigma_b$ be a quasiconformal map. The change of base point map α^* is defined by

$$\begin{aligned} \alpha^* : \widetilde{T}_{\mathrm{WP}}(\Sigma_b) &\longrightarrow \widetilde{T}_{\mathrm{WP}}(\Sigma_a) \\ [\Sigma_b, g, \Sigma_1, \phi] &\longmapsto [\Sigma_a, g \circ \alpha, \Sigma_1, \phi] \end{aligned} \tag{4.7}$$

This is completely analogous to the usual change of base point biholomorphism for the Teichmüller space $T(\Sigma)$ (see the paragraph following Theorem 3.12). From the general definition of the Schiffer variation map in (3.3), it is worth noting that the coordinates for $\widetilde{T}_{\mathrm{WP}}(\Sigma_{\mathrm{WP}})$ as defined in (3.7) actually have this change of base point biholomorphism built in. From this observation we easily obtain the following theorem.

Theorem 4.19. *The change of base point map in (4.7) is a biholomorphism.*

Proof. The map α^* has inverse $(\alpha^*)^{-1} = (\alpha^{-1})^*$ and hence is a bijection. Consider the points $p = [\Sigma_b, g, \Sigma_1, \phi]$ and $q = \alpha^*(p) = [\Sigma_a, g \circ \alpha, \Sigma_1, \phi]$. One can choose coordinates, as in equation (3.7), for neighborhoods of p and q which use the same Schiffer variation on Σ_1, and thus the same map ν^ϵ. In terms of these local coordinates, the map α^* is the identity map and so is certainly holomorphic. The same argument shows that $(\alpha^{-1})^*$ is holomorphic and hence α^* is biholomorphic.

□

The next task is to relate the preceding change of base point map to the one between bordered surfaces. Let Σ_b^B and Σ_a^B be bordered Riemann surfaces of type (g, n) and fix riggings $\tau_b \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma_b^B)$ and $\tau_a \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma_a^B)$. Then there exists $\rho \in \mathrm{QC}_0(\Sigma_a^B, \Sigma_b^B)$ such that $\rho \circ \tau_a = \tau_b$. In fact one can prove a stronger statement [21, Corollary 4.7 and Lemma 4.17]: Given any quasiconformal map $\rho' : \Sigma_a^B \to \Sigma_b^B$, there exists $\rho \in \mathrm{QC}_0(\Sigma_a^B, \Sigma_b^B)$ such that $\rho \circ \tau_a = \tau_b$ and ρ is homotopic (not rel boundary) to ρ'. The map ρ' is obtained by deforming ρ in a neighborhood of the boundary curves so as to have the required boundary values.

For such a ρ, define the change of base point map

$$\begin{aligned} \rho^* : T^B_{\mathrm{WP}}(\Sigma_b^B) &\longrightarrow T^B_{\mathrm{WP}}(\Sigma_a^B) \\ [\Sigma_b^B, f, \Sigma_1^B] &\longmapsto [\Sigma_a^B, f \circ \rho, \Sigma_1^B] \end{aligned} \tag{4.8}$$

which is just the usual change of base point map restricted to the WP-class Teichmüller space, together with the added condition of compatibility with the fixed base riggings.

Let Σ_b and Σ_a be the punctured surfaces obtained from Σ_b^B and Σ_a^B by sewing on caps via τ_b and τ_a respectively. Given ρ as above we have its quasiconformal extension $\tilde{\rho} : \Sigma_a \to \Sigma_b$ defined by

$$\tilde{\rho} = \begin{cases} \rho, & \text{on } \overline{\Sigma_a^B} \\ \mathrm{id}, & \text{on } \mathbb{D} \end{cases}$$

as in (4.5). Let $\tilde{\rho}^*$ be the change of base point biholomorphism as in (4.7).

Lemma 4.20. *Let (Σ_b^B, τ_b), (Σ_a^B, τ_a), ρ, ρ^*, $\tilde{\rho}$ and $\tilde{\rho}^*$ be as above. Then the diagram*

$$\begin{array}{ccc} T^B_{\mathrm{WP}}(\Sigma_b^B) & \xrightarrow{\rho_*} & T^B_{\mathrm{WP}}(\Sigma_a^B) \\ \Pi_{\mathrm{WP}} \downarrow & & \downarrow \Pi_{\mathrm{WP}} \\ \widetilde{T}_{\mathrm{WP}}(\Sigma_b) & \xrightarrow{\tilde{\rho}^*} & \widetilde{T}_{\mathrm{WP}}(\Sigma_a) \end{array}$$

commutes.

Proof. Let $[\Sigma_b^B, f, \Sigma_1^B] \in T^B_{\mathrm{WP}}(\Sigma_b^B)$. We have that

$$\begin{aligned} \Pi_{\mathrm{WP}} \circ \rho^*([\Sigma_b^B, f, \Sigma_1^B]) &= \Pi_{\mathrm{WP}}([\Sigma_a^B, f \circ \rho, \Sigma_1^B]) \\ &= [\Sigma_a^B \#_{\tau_a} \mathbb{D}, \widetilde{f \circ \rho}, \Sigma_1^B \#_{f\circ\rho\circ\tau_a} \mathbb{D}, f \widetilde{\circ \rho \circ \tau_a}] \\ &= [\Sigma_a^B \#_{\tau_a} \mathbb{D}, \widetilde{f \circ \rho}, \Sigma_1^B \#_{f\circ\tau_b} \mathbb{D}, \widetilde{f \circ \tau_b}] \\ &= [\Sigma_a, \widetilde{f \circ \rho}, \Sigma_1, \widetilde{f \circ \tau_b}] \end{aligned}$$

since $\rho \circ \tau_a = \tau_b$. On the other hand

$$\tilde{\rho}^* \circ \Pi_{\mathrm{WP}}([\Sigma_b^B, f, \Sigma_1^B]) = \tilde{\rho}^*([\Sigma_b^B \#_{\tau_b} \mathbb{D}, \tilde{f}, \Sigma_1^B \#_{f\circ\tau_b} \mathbb{D}, \widetilde{f \circ \tau_b}]) = [\Sigma_b, \tilde{f} \circ \tilde{\rho}, \Sigma_1, \widetilde{f \circ \tau_b}].$$

The claim follows from the fact that $\widetilde{f \circ \rho} = \tilde{f} \circ \tilde{\rho}$ (Remark 4.10). □

Theorem 4.18, Theorem 4.19, and Lemma 4.20 immediately imply the following theorem.

Theorem 4.21. *Let (Σ_b^B, τ_b) and (Σ_a^B, τ_a) be a pair of rigged bordered Riemann surfaces, with $\tau_b \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma_b^B)$ and $\tau_a \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma_a^B)$. Let $\rho \in \mathrm{QC}_0(\Sigma_1^B, \Sigma_b^B)$ satisfy $\rho \circ \tau_a = \tau_b$. Then the change of base point map ρ^* given by equation (4.8) is a biholomorphism.*

Corollary 4.22. *The complex Hilbert manifold structure on $T_{\mathrm{WP}}(\Sigma^B)$ is independent of the choice of rigging $\tau \in \mathrm{QS}_{\mathrm{WP}}(\Sigma^B)$.*

Proof. Apply Theorem 4.21 with $\Sigma_b^B = \Sigma_a^B = \Sigma^B$. □

This immediately implies

Corollary 4.23. *Let $\rho \in \mathrm{QC}_0(\Sigma_a^B, \Sigma_b^B)$. The change of base point map $\rho^* : T^B_{\mathrm{WP}}(\Sigma_b^B) \longrightarrow T^B_{\mathrm{WP}}(\Sigma_a^B)$ is a biholomorphism.*

Finally, the following theorem shows that the complex structure of the WP-class Teichmüller space is compatible with the standard complex structure.

Theorem 4.24. *The inclusion map from $T_{\mathrm{WP}}(\Sigma^B)$ to $T(\Sigma^B)$ is holomorphic.*

Proof. Since Π has local holomorphic inverses the inclusion map from $T_{\mathrm{WP}}(\Sigma^B)$ to $T(\Sigma^B)$ can be locally written as $\Pi^{-1} \circ \iota \circ \Pi_{\mathrm{WP}}$ where $\iota : \widetilde{T}_{\mathrm{WP}}(\Sigma) \to \widetilde{T}(\Sigma)$ is inclusion. The theorem follows from the facts that Π^{-1} and Π_{WP} are holomorphic and ι is holomorphic by Theorem 3.30. □

4.4. **Rigged moduli space is a Hilbert manifold.** In this section we show that the rigged moduli space of conformal field theory originating with Friedan and Shenker [8], with riggings chosen as in this paper, have Hilbert manifold structures.

First we define the moduli spaces. There are two models, which we will refer to as the border and the puncture model. These models are defined as follows:

Definition 4.25. Fix non-negative integers g and n, such that $2g-2+n>0$.

(1) The border model of the WP-class rigged moduli space is

$$\mathcal{M}^B_{\mathrm{WP}}(g,n) = \{(\Sigma^B,\psi) \,:\, \Sigma^B \text{ bordered of type } \;(g,n),\quad \psi\in \mathrm{Rig}_{\mathrm{WP}}(\Sigma^B)\}/\sim$$

where $(\Sigma^B_1,\psi)\sim(\Sigma^B_2,\phi)$ if and only if there is a biholomorphism $\sigma:\Sigma^B_1\to\Sigma^B_2$ such that $\phi=\sigma\circ\psi$.

(2) The puncture model of the rigged moduli space is

$$\mathcal{M}^P_{\mathrm{WP}}(g,n) = \{(\Sigma,\psi) \,:\, \Sigma \text{ punctured of type } (g,n),\quad \psi\in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}(\Sigma)\}/\sim$$

where $(\Sigma_1,\psi)\sim(\Sigma_1,\phi)$ if and only if there is a biholomorphism $\sigma:\Sigma_1\to\Sigma_2$ such that $\phi=\sigma\circ\psi$.

The puncture and border models (but with different classes of riggings) were used by [31] and [27] respectively, in the study of conformal field theory. It was understood from their inception that these rigged moduli spaces are in bijective correspondence, as can be seen by cutting and sewing caps. However, one needs to be careful about the exact classes of riggings used to make this statement precise. Replacing "bijection" with "biholomorphism" in this statement of course requires the careful construction of a complex structure on at least one of these spaces. It was shown in [21] that these two moduli spaces are quotient spaces of $T(\Sigma^B)$ by a fixed-point-free properly discontinuous group, and thus inherit a complex Banach manifold structure from $T(\Sigma^B)$. Similarly, we will demonstrate that the WP-class rigged moduli spaces inherits a complex Hilbert manifold structure from $T_{\mathrm{WP}}(\Sigma^B)$. We first need to show that the action of $\mathrm{PModI}(\Sigma^B)$ defined by (4.1) is fixed point free and properly discontinuous.

Theorem 4.26. *The modular group* $\mathrm{PModI}(\Sigma^B)$ *acts properly discontinuously and fixed-point-freely on* $T_{\mathrm{WP}}(\Sigma^B)$. *The action of each element of* $\mathrm{PModI}(\Sigma^B)$ *is a biholomorphism of* $T_{\mathrm{WP}}(\Sigma^B)$.

Proof. Recall that $\mathrm{DB}(\Sigma^B)$ preserves $T_{\mathrm{WP}}(\Sigma^B)$ by Lemma 4.4. By [21, Lemma 5.2], $\mathrm{DB}(\Sigma^B)$ acts properly discontinuously and fixed-point freely on $T(\Sigma^B)$. Thus $\mathrm{DB}(\Sigma^B)$ acts fixed-point freely on $T_{\mathrm{WP}}(\Sigma^B)$. Now let $x\in T_{\mathrm{WP}}(\Sigma^B)$. There is a neighborhood U of x in $T(\Sigma^B)$ such that $[\rho]U\cap U$ is empty for all $[\rho]\in\mathrm{DB}(\Sigma^B)$. Clearly $V=U\cap T_{\mathrm{WP}}(\Sigma^B)$ has the same property, and is open in $T_{\mathrm{WP}}(\Sigma^B)$ by Theorem 4.24.

Each element $[\rho]\in\mathrm{PModI}(\Sigma^B)$ is a biholomorphism of $T_{\mathrm{WP}}(\Sigma^B)$, by observing that $\rho\circ\tau=\tau$ and applying Theorem 4.21. □

We now show that the rigged moduli spaces are Hilbert manifolds. Let Σ^B be a fixed bordered Riemann surface of type (g,n) and let $\tau\in\mathrm{Rig}(\Sigma^B)$ be a fixed rigging. Define the mapping

$$\begin{aligned} P : T(\Sigma^B) &\longrightarrow \mathcal{M}^B(g,n) \\ [\Sigma^B,f,\Sigma^B_1] &\longmapsto (\Sigma^B_1, f\circ\tau) \end{aligned}$$

where $f \circ \tau = (f \circ \tau_1, \ldots, f \circ \tau_n)$. Note that this map depends on the choice of Σ^B and τ. If we choose $\tau \in \mathrm{Rig}_{\mathrm{WP}}(\Sigma^B)$, we have the map

$$P_{\mathrm{WP}} = P|_{T_{\mathrm{WP}}(\Sigma^B)}.$$

It follows immediately from Proposition 4.8 that P_{WP} maps into $\mathcal{M}^B_{\mathrm{WP}}(g,n)$.

Theorem 4.27. *Given any $p, q \in T_{\mathrm{WP}}(\Sigma^B)$, $P_{\mathrm{WP}}(p) = P_{\mathrm{WP}}(q)$ if and only if $q = [\rho]p$ for some $[\rho] \in \mathrm{PModI}(\Sigma^B)$. Moreover, P_{WP} is a surjection onto $\mathcal{M}^B_{\mathrm{WP}}(g,n)$.*

Proof. All of these claims hold in the non-WP setting by [21, Theorem 5.2]. Thus the first claim follows immediately. It was already observed that π_{WP} maps into $\mathcal{M}^B_{\mathrm{WP}}(g,n)$. To show that π_{WP} is surjective, observe that by [21, Theorem 5.2], for any $[\Sigma^B_1, \psi] \in \mathcal{M}^B_{\mathrm{WP}}(g,n)$ there is a $[\Sigma^B, f_1, \Sigma^B_*] \in T(\Sigma^B)$ such that $[\Sigma^B_*, f_1 \circ \tau] = [\Sigma^B_1, \psi]$. By composing with a biholomorphism we can assume that $\Sigma^B_* = \Sigma^B_1$ and $f_1 \circ \tau = \psi$. Thus $f_1 = \psi \circ \tau^{-1}$. Since for $i = 1, \ldots, n$ we have $\psi_i \circ \tau_i^{-1} \in \mathrm{QS}_{\mathrm{WP}}(\partial_i \Sigma^B, \partial_i \Sigma^B_1)$ by Proposition 2.17, $f_1 \in \mathrm{QC}_0(\Sigma^B, \Sigma^B_1)$. Thus $[\Sigma^B, f_1, \Sigma^B_1] \in T_{\mathrm{WP}}(\Sigma^B)$ and $P_{\mathrm{WP}}([\Sigma^B, f_1, \Sigma^B_1]) = [\Sigma^B_1, \psi]$, which completes the proof. □

This shows that $T_{\mathrm{WP}}(\Sigma^B)/\,\mathrm{PModI}(\Sigma^B)$ and $\mathcal{M}^B_{\mathrm{WP}}(g,n)$ are bijective. They are also biholomorphic.

Corollary 4.28. *The rigged moduli space $\mathcal{M}^B(g,n)$ is a Hilbert manifold and the map P_{WP} is holomorphic and possesses local holomorphic inverses. The Hilbert manifold structure is independent of the choice of base surface Σ^B and rigging τ.*

Proof. This follows immediately from Theorem 4.27, the fact that $\mathrm{PModI}(\Sigma^B)$ acts fixed-point freely and properly discontinuously by biholomorphisms (Theorem 4.26), and the fact that the complex structure on $T_{\mathrm{WP}}(\Sigma^B)$ is independent of the choice of base rigging. □

It was shown in [21] that the border and puncture models of the rigged moduli space are in one-to-one correspondence, and that the puncture model can be obtained as a natural quotient of $\widetilde{T}_{\mathrm{WP}}(\Sigma)$. Those results pass immediately to the WP-class setting, with only very minor changes to the proofs (much as above). We will simply summarize the results here. Let Σ be a punctured Riemann surface of type (g,n). Denote by $\mathrm{PModP}(\Sigma)$ the modular group of quasiconformal maps $f : \Sigma \to \Sigma$ modulo the quasiconformal maps homotopic to the identity rel boundary. Elements $[\rho]$ of $\mathrm{PModP}(\Sigma)$ act on $\tilde{T}_{\mathrm{WP}}(\Sigma)$ via $[\rho][\Sigma, f_1, \Sigma_1, \psi] = [\Sigma, f_1 \circ \rho, \Sigma_1, \psi]$. Define the projection map

$$\begin{aligned} Q : \widetilde{T}_{\mathrm{WP}}(\Sigma) &\to \mathcal{M}^P_{\mathrm{WP}}(g,n) \\ [\Sigma, f, \Sigma_1, \psi] &\mapsto [\Sigma_1, \psi]. \end{aligned}$$

Finally, define the map

$$\begin{aligned} \mathcal{I} : \mathcal{M}^P(g,n) &\to \mathcal{M}^B(g,n) \\ [\Sigma, \phi] &\mapsto [\Sigma \backslash \overline{\phi_1(\mathbb{D}) \cup \cdots \cup \phi_n(\mathbb{D})}, \phi|_{\mathbb{S}^1}]. \end{aligned}$$

Theorem 4.29. *The moduli spaces $\mathcal{M}^P(g,n)$ and $\mathcal{M}^B(g,n)$ are in one-to-one correspondence under the bijection $\mathcal{I}$. Thus $\mathcal{M}^P(g,n)$ can be endowed with a unique Hilbert manifold structure so that $\mathcal{I}$ is a biholomorphism. The map Q satisfies*

(1) *$Q(p) = Q(q)$ if and only if there is a $[\rho] \in \mathrm{PModP}(\Sigma)$ such that $[\rho]p = [q]$*
(2) *Q is surjective,*
(3) *Q is holomorphic, and possesses a local holomorphic inverse in a neighborhood of every point.*

References

[1] Ahlfors, L. V. and Sario L. *Riemann Surfaces*, Princeton University Press, 1960.

[2] Astala, K.; and Zinsmeister, M. *Teichmüller spaces and BMOA*, Mathematische Annalen, **289** no 1 (1991), 613–625.

[3] Chae, S. B. *Holomorphy and Calculus in Normed Spaces*. With an appendix by Angus E. Taylor. Monographs and Textbooks in Pure and Applied Mathematics, 92. Marcel Dekker, Inc., New York, 1985.

[4] G. Cui, *Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces*, Sci. China Ser. A 43 (2000), no. 3, 267–279.

[5] Cui, G. and Zinsmeister, M. *BMO-Teichmüller spaces*. Illinois Journal of Mathematics **48** no. 4 (2004), 1223–1233.

[6] Earle, C. J. and Fowler, R. S. *Holomorphic families of open Riemann surfaces*, Math. Ann. **270** (1985), no. 2, 249–273.

[7] Fritzsche, K. and Grauert, H. *From Holomorphic Functions to Complex Manifolds*, Graduate Texts in Mathematics, **213**, Springer-Verlag, New York, 2002.

[8] Friedan, D. and Shenker, S. *The analytic geometry of two-dimensional conformal field theory*. Nuclear Physics B **281** (1987), 509–545.

[9] Gardiner, F. P. *Schiffer's interior variation and quasiconformal mapping*, Duke Math. J. **42** (1975), 371–380.

[10] Gardiner, F. P. and Sullivan, D. P. *Symmetric structures on a closed curve*. American Journal of Mathematics, **114** no. 4 (1992), 683–736.

[11] Hui, G. *Integrable Teichmüller spaces*. Science in China, **43** no 1, Series A (2000), 47–58.

[12] Huang, Y.-Z. *Two-dimensional Conformal Geometry and Vertex Operator Algebras*. Progress in Mathematics, **148**. Birkhäuser Boston, Inc., Boston, MA, 1997.

[13] Huang, Y.-Z. and Kong, L. *Modular invariance for conformal full field algebras*, Trans. Amer. Math. Soc. **362** no.6, (2010), 3027–3067.

[14] Hubbard, J. H., *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*, Vol. 1, Teichmüller theory, Matrix Editions, Ithaca, NY, 2006.

[15] Lehto, O. and Virtanen, K. I. *Quasiconformal Mappings in the Plane*. 2nd edition. Die Grundlehren der mathematischen Wissenshcaften, Band 126. Springer-Verlag, New York-Heidelberg, 1973.

[16] Mujica, J. *Complex Analysis in Banach Spaces*. North Holland, 1986.

[17] Nag, S, *Schiffer variation of complex structure and coordinates for Teichmüller spaces*, Proc. Indian Acad. Sci. Math. Sci. **94** no. 2-3, (1985), 111–122.

[18] Nag, S. *The Complex Analytic Theory of Teichmüller Spaces*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1988.

[19] Nag, S. and Verjovsky, A. Diff($\mathbb{S}^1$) *and the Teichmüller spaces*, Comm. Math. Phys. **130** (1990), no. 1, 123–138.

[20] Radnell, D. *Schiffer Variation in Teichmüller Space, Determinant Line Bundles and Modular Functors*, PhD Thesis, Rutgers University, New Brunswick, NJ, 2003.

[21] Radnell, D. and Schippers, E. *Quasisymmetric sewing in rigged Teichmüller space*, Commun. Contemp. Math. **8** no 4 (2006), 481–534.

[22] Radnell, D. and Schippers, E., *A complex structure on the set of quasiconformally extendible non-overlapping mappings into a Riemann surface*, J. Anal. Math. **108** (2009), 277–291

[23] Radnell, D. and Schippers, E., *Fiber structure and local coordinates for the Teichmüller space of a bordered Riemann surface*, Conform. Geom. Dyn. **14** (2010), 14–34.

[24] Radnell, D.; Schippers, E. and Staubach, Wolfgang. *Weil-Petersson class non-overlapping mappings into a Riemann surface*. Preprint 2014.

[25] Radnell, D., Schippers, E., and Staubach, W. *A Hilbert manifold structure on the refined Teichmüller space of bordered Riemann surfaces*, Preprint, arXiv:1207.0973.

[26] Radnell, D.; Schippers, E. and Staubach, Wolfgang. *A convergent Weil-Petersson metric on the Teichmüller space of bordered Riemann surfaces*. Preprint, arXiv:1403.0868.

[27] Segal, G. *The definition of conformal field theory*, Topology, Geometry and Quantum Field Theory (U. Tillmann, ed.), London Mathematical Society Lecture Note Series, vol. 308, Cambridge University Press, 2004, Original preprint 1988, pp. 421–576.
[28] Y. Shen, Weil-Petersson Teichmüller space, prepint, arXiv:1304.3197.
[29] Tatkhajan, L. A. and Teo, L.-P. *Weil-Petersson Metric on the Universal Teichmüller Space.* Memoirs of the American Mathematical Society. **183** no 861 (2006).
[30] Teo, L.-P. *Velling-Kirillov metric on the universal Teichmüller curve.* J. Anal. Math. **93** (2003), 271–307.
[31] Vafa, C. *Conformal theories and punctured surfaces*, Phys. Lett. B, **199** no. 2, (1987), 195–202.

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