The Loewner and Hadamard variations

Oliver Roth[1] Eric Schippers[2]

[1]Department of Mathematics
University of Würzburg
Würzburg, Germany

[2]Department of Mathematics
University of Manitoba
Winnipeg, Canada

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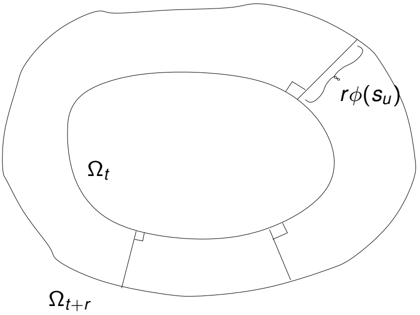
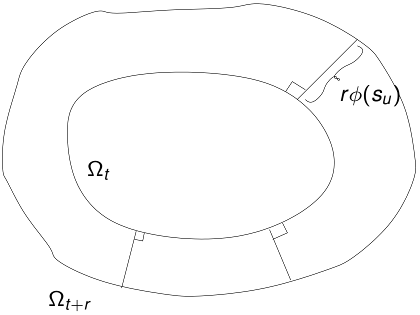
Hadamard variation



Hadamard variation



g_t = Green's function, n = outward normal, s = arc length

The formula:

$$\frac{dg_t}{dt}(z,\zeta) = \frac{1}{2\pi}\int_{\partial\Omega_t} \frac{\partial g_t}{\partial n_u}(u,z)\frac{\partial g}{\partial n_u}(u,\zeta)\phi(s_u)ds_u.$$

Advantages and drawbacks

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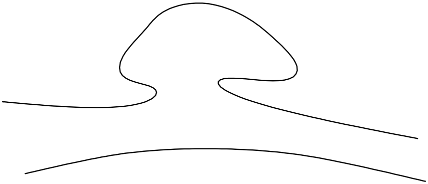

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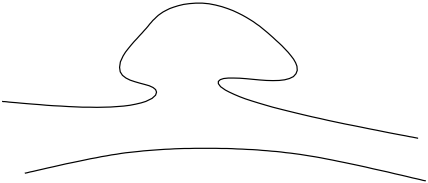
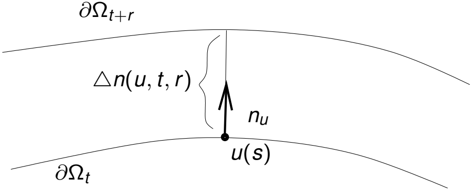


Motivation for getting "all" of them:

- Can prove monotonicity theorems: $D_1 \subset D_2 \Rightarrow I(D_1) \subset I(D_2)$.

Won't pursue monotonicity theorems in this talk.

An improvement of Hadamard formula



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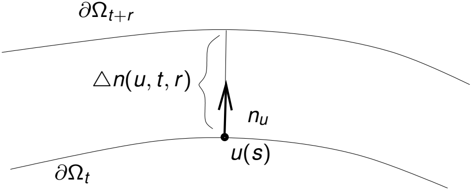

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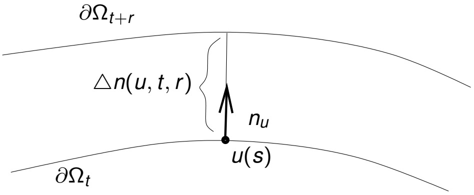

$$\nu_t(u) := \frac{d}{dr}\Big|_{r=0} \triangle n(u,t,r)$$

Theorem (S 2004)

Let Ω_t, $t \in (a,b)$ be a family of domains, bounded by C^2 curves such that $\Omega_s \subset \Omega_t$ whenever $s \leq t$ and the curves describe a C^2 injective homotopy. Then

$$\frac{dg_t}{dt}(z,\zeta) = \frac{1}{2\pi}\int_{\partial\Omega_t} \frac{\partial g_t}{\partial n_u}(u,z)\frac{\partial g}{\partial n_u}(u,\zeta)\nu_t(u)ds_u.$$

Loewner chains

Definition

A Loewner chain is a one-parameter family of holomorphic maps $f_t : \mathbb{D} \to \mathbb{C}$:

1. $f_t(0) = 0$, $f_t'(0) = e^t$
2. f_t one-to-one
3. $t \leq s \Rightarrow f_t(\mathbb{D}) \subset f_s(\mathbb{D})$

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Infinitesimal generators of Loewner chains:
$\mathcal{P} = \{p : \mathbb{D} \to \mathbb{C},\ p(0) = 1\}$
Loewner chains satisfy the Loewner PDE: $\exists p_t \in \mathcal{P}$, measurable in t, such that a.e. in t

$$\frac{df_t}{dt}(z) = zp_t(z)f_t'(z).$$

The Problem

Every smooth injective homotopy of simply connected domains is a Loewner chain (perhaps after reparameterization).

Conversely $\mathcal{P}$ should generate all outward directions.

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- Relate the Loewner infinitesimal generator p_t to Hadamard infinitesimal generator ν_t.
- For a fixed initial domain $D = f(\mathbb{D})$, can you move in any direction $p \in \mathcal{P}$?

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Heins[71] derived a Loewner equation for Green's function.

The Solution

Notation:

- f_t a Loewner chain, $p_t \in \mathcal{P}$ the infinitesimal generator
- $u(s)$ parametrizes the boundary of $f_t(\partial\mathbb{D})$ w.r.t. arc length
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Theorem (Roth and S, 08)

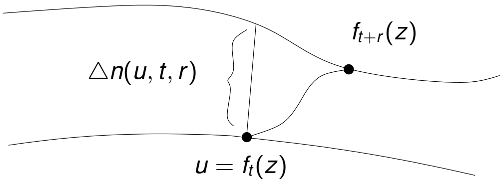
Let f_t be a Loewner chain, satisfying

$$\dot{f}_t = z p_t f_t'(z)$$

such that $F(t,s) = f_t(e^{is})$ is a C^2 injective homotopy of closed curves.

$$\nu_t(u) = -Re\left(\frac{1}{i} f_t^{-1}(u) f' \circ f_t^{-1}(u) p_t \circ f_t^{-1}(u) \frac{d\bar{u}}{ds}\right).$$

A picture of the proof



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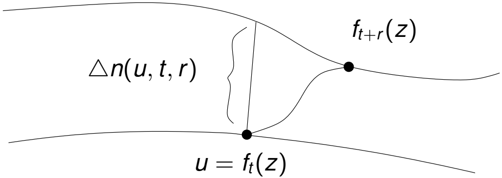

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Remark:

- The expression $f_t^{-1} f' \circ f_t^{-1} p_t \circ f_t^{-1}$ is $Ad_f\, zp$ in some sense.
- If $p_t = p$ is constant and $f_0(z) = z$, then $f_t^{-1} f' \circ f_t^{-1} p_t \circ f_t^{-1} = p$

Existence theorem

Theorem (Roth and S, 2008)

Let f be 1-1, p satisfy the following conditions:

- $f : \mathbb{D} \to D_0$, $f \in C^3(\overline{\mathbb{D}})$
- $p \in \mathcal{P} \cap C^2(\overline{\mathbb{D}})$

There exists a Loewner chain f_t on $[0, T]$ satisfying $f_0 = f$, $\dot{f}_t(z) = zp_t(z)f_t'(z)$ and $p_0 = p$.

Proof.

Use a normal variation $(s, r) \mapsto u(s) + \nu(u(s))r$ at each point u on the boundary, where $\nu(u)$ is chosen to generate p. □

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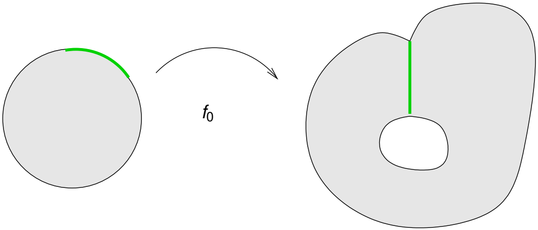
Question: How much can you weaken the conditions?

Some examples

Example: for some f_0 there is no Loewner chain f_t



Remark: There may be a solution that continues to be analytic on the disc, but not univalent.

Question: Is there a Loewner chain with $f_0(z) = z$ and for $|\kappa| = 1$

$$p_0 = \frac{1+\kappa}{1-\kappa}?$$

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$$p_t = \frac{1 + e^{-3t} z}{1 - e^{-3t} z}$$

the solution to $\dot{f}_t(z) = z p_t(z) f_t'(z)$ with $f_0(z) = z$ is

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Not analytic on $\mathbb{D}$ for any $t > 0$!

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Asymmetry: there is always a solution to $\dot{f}_t(z) = -zp_t(z)f_t'(z)$, $f_0(z) = z$.

Loewner ordinary differential equation

A naive attempt at a proof of existence theorem

- Given p, solve $\dot{w}_{s,t} = -w_{s,t}\, p_t \circ w_{s,t}$ with $p_0 = p$, $w_{s,s}(z) = z$

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Corollary (Roth and S, 2008)

For $f : \mathbb{D} \to \mathbb{C}$ and $p \in \mathcal{P}$ which are sufficiently smooth, there exists a p_t, solution to the Loewner ODE

$$\dot{w}_t = -w_t\, p_t \circ w_t$$

on $[0, \infty)$ such that $w_0(z) = z$, $p_0 = p$ and $\lim_{t\to\infty} e^t w_t = f$.

Relation with Heins' Loewner equation for Green's function

Recall Heins derived a Loewner equation for Green's function.
What's the relation of Heins' equation with generalized Hadamard variation?

- Heins' Loewner equation for Green's function is recovered from the generalized Hadamard formula

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- You can generalize the (generalized) Hadamard formula to any Loewner chain (i.e. remove smoothness)
- In fact can let $\nu = d\mu$ for some μ increasing and of bounded variation in Hadamard variational formula for inward variations.

Loewner PDE for any subordination chain

Finally, can prove Loewner's equation for arbitrary subordination chains - no smoothness assumption on $f_t'(0)$

Theorem (Roth and S, 2008)

Let $f_t : \mathbb{D} \to \mathbb{C}$, $t \in [0, T]$, $f_t(0) = 0$, be univalent maps satisfying $s < t \Rightarrow f_s(\mathbb{D}) \subset f_t(\mathbb{D})$. There is a p_t such that $Re(p) > 0$, measurable in t, such that

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Proof.

- Use Hein's idea of deriving the Loewner equation from a Loewner equation for Green's function.
- Green's function is monotonic, and hence differentiable a.e.

□