The Neretin-Segal semigroup and the Teichmüller space of an annulus

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Some Context

- Physically: conformal field theories are certain statistical and quantum field theories, invariant under local rescalings and rotations.
- Mathematically: major research problem. Sketch of rigorous mathematical model given by G. Segal in infamous preprint.
- David Radnell and I are doing “contract work” on analytic problems in realizing this definition.
- This leads to insight in function theory.

Outline

- Define the Neretin-Segal semigroup and its multiplication (precise).
- Prove some theorems about a complex structure of this semigroup.
- Relate it to a Teichmüller space.
- Discuss the multiplication.

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This is part of a more general program with David Radnell, relating moduli spaces in conformal field theory to quasiconformal Teichmüller spaces.

CFT/GFT

Some work in the intersection of conformal field theory and geometric function theory:

- Nag and Sullivan (95).
- Neretin (89 –)
- Takhtajan and Teo (2006–)
- Radnell and S (2006–)
- Markina, Prokhorov, Vasil'ev (2006–)

Motivation for the definition of Neretin-Segal semigroup

Let $\mathrm{Diff}(S^1)$ denote the set of orientation-preserving diffeomorphisms of the circle. Its Lie algebra is:

$$\text{span}\left\{ \frac{\partial}{\partial\theta},\ \cos n\theta \frac{\partial}{\partial\theta},\ \sin n\theta \frac{\partial}{\partial\theta} : n \in \mathbb{Z} \right\}$$

whose complexification is the “Witt algebra”

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Theorem (Lempert, 97)

There is no Lie group whose Lie algebra is the Witt algebra. $\mathrm{Diff}(S^1)$ *has no complexification.*

Neretin-Segal semigroup

Consider triples (A, ϕ_1, ϕ_2) such that

- A is an annulus
- $\phi_1 : S^1 \to \partial_1 A$ is a quasisymmetry
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- There is a conformal map $\sigma : A \to B$ such that
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Definition

The Neretin-Segal semigroup is:

$$\{(A, \phi_1, \phi_2)\}/\sim .$$

Some remarks

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- $\phi_i : S^1 \to \partial_i A$ called "riggings"
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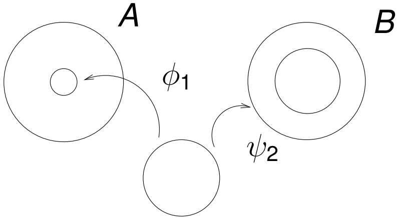
- Quasisymmetries are not the usual choice for riggings in conformal field theory (Neretin and Segal have analytic diffeomorphisms).
- It is *necessary* to use quasisymmetries, to get a connection with Teichmüller theory.
- Some do use quasisymmetric boundary parametrizations. (Pickrell [98] (Neretin-Segal semigroup); also Nag and Sullivan [95], Takhtajan and Teo [06])

Multiplying rigged annuli

Sewing:

Given (A, ϕ_1, ϕ_2) and (B, ψ_1, ψ_2), identify points on the boundaries $\partial_1 A$ and $\partial_2 B$ via $\psi_2 \circ \phi_1^{-1}$ to get $A \# B$.



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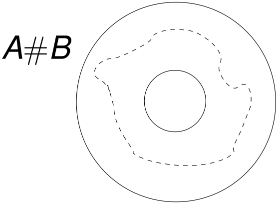


Definition

The multiplication is

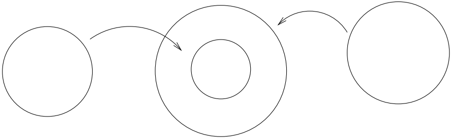
$$[A, \phi_1, \phi_2] \times [B, \psi_1, \psi_2] = [\sigma(A \# B), \sigma \circ \psi_1, \sigma \circ \phi_2].$$

where σ uniformizes $A \# B$ to an annulus.

Another model of the Neretin-Segal semigroup

Sewing on caps:

Given (A, ϕ_1, ϕ_2), use ϕ_i to identify points of $\partial\mathbb{D}$ with $\partial_i A$, $i = 1, 2$. Sewing on two copies of $\mathbb{D}$ results in a Riemann surface biholomorphically equivalent to a sphere.



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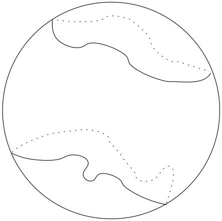

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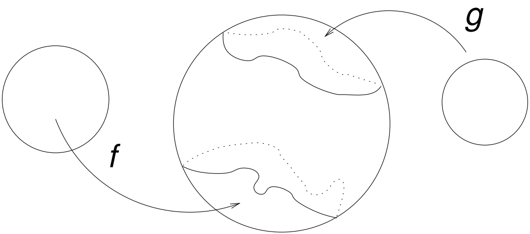

And we can keep track of the riggings.

The non-overlapping mapping model

Let

$$\mathbb{D} = \{z : |z| < 1\} \quad \text{and} \quad \mathbb{D}^* = \{z : |z| < 1\} \cup \{\infty\}.$$

Definition

Let $\mathcal{A}^0 = \{(f, g)\}$ where $f : \mathbb{D} \to \mathbb{C}$, $g : \mathbb{D}^* \to \overline{\mathbb{C}}$ are one-to-one holomorphic maps satisfying

1. f has a quasiconformal extension to $\mathbb{C}$ and g has a quasiconformal extension to $\overline{\mathbb{C}}$.
2. $f(\overline{\mathbb{D}}) \cap g(\overline{\mathbb{D}^*}) = \emptyset$
3. $f(0) = 0$
4. $g(\infty) = \infty$, $g'(\infty) = 1$

A complex structure on the Neretin-Segal semigroup

Let

$$A_1^\infty = \{h : \mathbb{D} \to \mathbb{C} : h \text{ holomorphic}, \sup_{z \in \mathbb{D}} (1 - |z|^2)|h(z)| < \infty\}.$$

Define $\Psi(f) = f''/f'$ and $\iota(z) = 1/z$.

$$\begin{aligned} B : \mathcal{A}^o &\to A_1^\infty \oplus \mathbb{C} \oplus A_1^\infty \oplus \mathbb{C} \\ (f, g) &\mapsto \left(\Psi(f), f'(0), \Psi(\iota \circ g \circ \iota), g'(\infty)\right). \end{aligned}$$

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Theorem (Radnell & S)

$B(\mathcal{A}^0)$ is a complex submanifold of $A_1^\infty \oplus \mathbb{C} \oplus A_1^\infty \oplus \mathbb{C}$. Thus $\mathcal{A}^o$ possesses a complex structure.

Elements of proof

- If we enlarge $\mathcal{A}^0$ by removing the condition that $g'(\infty) = 1$, then the image under B is open.
- This is a special case of a general result on non-overlapping mappings into a Riemann surface (Radnell and S, Journal d'Analyse 2009).
- This requires equicontinuity of families of quasiconformal maps
- It also requires some results of Lee-Peng Teo, on the compatibility of the pre-Schwarzian embedding with the Schwarzian embedding.
- Imposing the condition that $g'(\infty) = 1$ results in a complex submanifold (easy).

The more general result

- Let Σ^B be a Riemann surface of genus g with n boundary curves homeomorphic to S^1.
- Let Σ^P be a punctured Riemann surface obtained by sewing on n copies of the punctured disc.

Theorem (Radnell & S, Journal d'Analyse, 2009)

The set of non-overlapping, quasiconformally extendible conformal mappings from the punctured disc into Σ^P, taking punctures to punctures has a natural complex structure.

Teichmüller space of annuli

Fix a doubly-connected Riemann surface A. Consider triples (A, f, A_1) where $f : A \to A_1$ is quasiconformal, A and A_1 are doubly-connected Riemann surfaces.

Definition

The Teichmüller space of annuli is

$$T(A) = \{(A, f, A_1)\}/ \sim$$

where $(A, f_1, A_1) \sim (A, f_2, A_2)$ if there is a biholomorphism $\sigma : A_1 \to A_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity via a homotopy which is constant on ∂A.

Relation between Neretin-Segal semigroup and $T(A)$

Theorem (Radnell & S)

The Neretin-Segal semigroup is in one-to-one correspondence with $T(A)/\mathbb{Z}$. The $\mathbb{Z}$-action is by a properly-discontinuous fixed-point free group of biholomorphisms.

Note: this cannot be proven without extending the riggings to quasisymmetries.

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This is an exceptional case related to a general result [Radnell & S, 10], that identifies sets of non-overlapping mappings with fibers in Teichmüller space.

Sketch of the correspondence

- Fix a “base”: (A, τ^0, τ^∞) where
  - A is a doubly-connected subset of $\overline{\mathbb{C}}$, bordered by quasicircles
  - Quasisymmetric riggings τ_i extend to $\tilde{\tau}_i$ as follows:
  - $\tilde{\tau}^0 : \mathbb{D} \to \overline{\mathbb{C}}$ is conformal with q.c. extension and maps onto bounded component of A^c
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- Every element of $T(A)$ has a *canonical representative* (A, h, A') where
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There is a $\mathbb{Z}$-action which preserves boundary values.
Two canonical representatives of $T(A)$ are equivalent in $T(A)/\mathbb{Z}$ if and only if their h's have the same boundary values.

Correspondence continued

The correspondence is:

- Each element of $T(A)$ represented by a map $\tilde{h} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$
- $\tilde{h} \mapsto (\tilde{h} \circ \tilde{\tau}^0, \tilde{h} \circ \tilde{\tau}^\infty)$.
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Surjective: every pair of riggings can be quasiconformally extended to an $\tilde{h}$ (by λ-lemma).
Injective: simple consequence of Teichmüller equivalence.

Compatibility

Theorem (Radnell & S)

The complex structure on $\mathcal{A}^o$ derived from $A_1^\infty \oplus \mathbb{C} \oplus A_1^\infty \oplus \mathbb{C}$ is compatible with the complex structure inherited from $T(A)$.

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Note similarity to Bers embedding.

Remarks about the proof

- We show that the bijection $T(A)/\mathbb{Z} \to \mathcal{A}^0$ is biholomorphic (not just Gâteaux holomorphic).
- A theorem of Chae says that you need only show it's Gâteaux holomorphic and locally bounded.
- Inverse function theorem for holomorphic maps does not hold in infinite dimensions.

A generic result

This theorem is an exceptional case, corresponding to a generic result.

Theorem (Radnell & S, Conformal Geometry and Dynamics, 2010)

Let Σ^B be a Riemann surface of genus g with n boundary curves homeomorphic to S^1. Let Σ^P be the punctured surface obtained by sewing on copies of the punctured disc. Assume that $2g - 2 + n > 0$.

- *The Teichmüller space $T(\Sigma^B)$ is fibered over $T(\Sigma^P)$, and the fibers are the set of non-overlapping mappings modulo a properly discontinuous group action.*
- *The complex structures of $T(\Sigma^B)$ and that of the non-overlapping mappings are compatible.*

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Remarks: The proof of the generic case was significantly more complicated.

On the other hand, the statement and proof of the exceptional case do not follow directly from the generic case.

Multiplication in $\mathcal{A}^0$

Theorem (Multiplication in non-overlapping mapping model)

Let $(f_1, g_1) \in \mathcal{A}$ and $(f_2, g_2) \in \mathcal{A}$ be two non-overlapping pairs representing rigged annuli.

$$(f_1, g_1) \cdot (f_2, g_2) = (F \circ f_2\,,\; G \circ g_1)\,.$$

where $F : \overline{g_2(\mathbb{D})}^c \to \overline{\mathbb{C}}$ and $G : \overline{f_1(\mathbb{D})}^c \to \overline{\mathbb{C}}$ are the unique normalized conformal maps such that

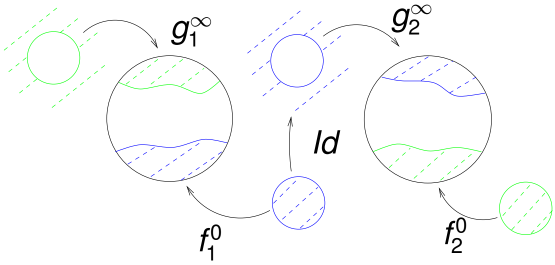
$$F^{-1} \circ G = g_2 \circ f_1{}^{-1}.$$

Proof.

Conformal welding. □

Explanation of multiplication

Sew with blue riggings:



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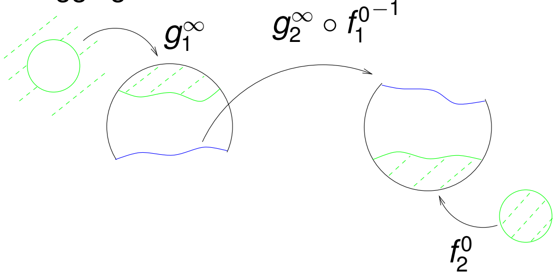

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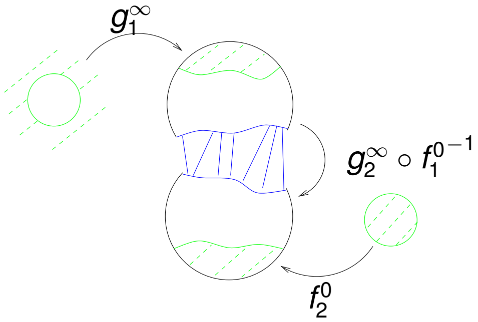
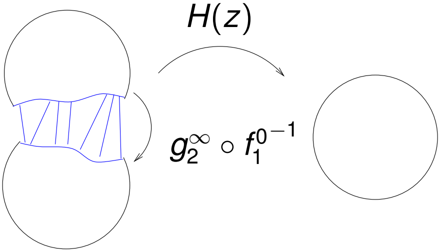


Explanation of multiplication: uniformize



$$H(z) = \begin{cases} G(z) & z \in \text{ top} \\ F(z) & z \in \text{ bottom} \end{cases}$$

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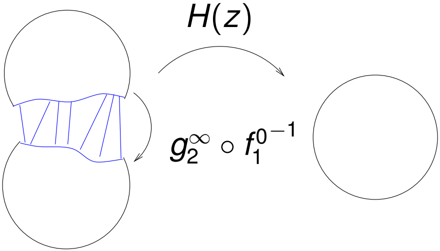

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$F \circ g_2^{\infty} \circ f_1^{\;-1} = G.$

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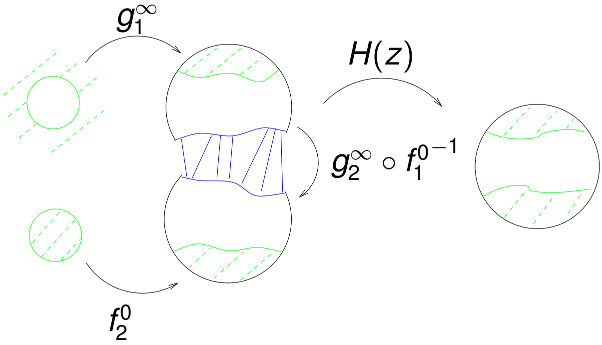

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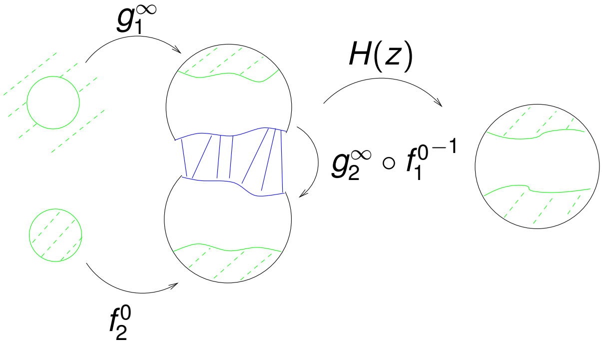

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Product: $(F \circ f_2, G \circ g_1)$

Attribution of the sewing equation

Sewing equation:

$$F \circ g_2 \circ {f_1}^{-1} = G.$$

The proof of the existence of a solution to the sewing equation is due to Y.-Z Huang, [97], for analytic parametrizations. (Largely algebraic.)

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Quasisymmetric case by Radnell & S; however the only change is the analytic condition. (Easy, using conformal welding).

Conformal welding was used in conformal field theory (unwittingly).

Multiplication is holomorphic

Theorem (Radnell & S)

Multiplication is holomorphic.

This is a special case of a more general theorem:

Theorem (Radnell & S, Communications in Contemporary Mathematics 2006)

The operation of sewing Riemann surfaces with boundary parametrizations is holomorphic (on an appropriate moduli space).

The subgroup of bounded univalent functions

Consider

$$\mathcal{E} = \{(f, \mathrm{Id}) \in \mathcal{A}^0\}.$$

- $f : \mathbb{D} \to \mathbb{D}$, is one-to-one, and quasiconformally extendible.
- This is a subgroup, with multiplication $(f_1, \mathrm{Id}) \times (f_2, \mathrm{Id}) = (f_1 \circ f_2, \mathrm{Id})$.

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So

$$(f_1, \mathrm{Id}) \cdot (f_2, \mathrm{Id}) = (F \circ f_2, G \circ \mathrm{Id}) = (f_1 \circ f_2, \mathrm{Id}).$$

$\mathcal{E}$ is a complex submanifold

Theorem (Radnell & S)

$\mathcal{E}$ is a complex submanifold of $\mathcal{A}^0$ (and thus of $T(A)/\mathbb{Z}$). Composition is holomorphic.

The subgroup of quasisymmetries

Extension $\mathcal{A}$ of $\mathcal{A}^0$: allow elements (f, g) where $f(\partial\mathbb{D})$ and $g(\partial\mathbb{D})$ overlap. i.e. possibly “degenerate” annuli .

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Extension $\mathcal{A}$ of $\mathcal{A}^0$: allow elements (f, g) where $f(\partial\mathbb{D})$ and $g(\partial\mathbb{D})$ overlap. i.e. possibly “degenerate” annuli .

We now include conformal welding pairs (f, g) corresponding to quasisymmetries $\phi : S^1 \to S^1$:

$$\phi = g^{-1} \circ f.$$

So $QS(S^1)$ is a subgroup of $\mathcal{A}$. The multiplication reduces to composition of quasisymmetries:

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So $QS(S^1)$ is a subgroup of $\mathcal{A}$. The multiplication reduces to composition of quasisymmetries:

Sewing equation says: $f_1 \circ g_2^{-1} = G^{-1} \circ F$
so the product $(f_1, g_1) \cdot (f_2, g_2) = (F \circ f_2, G \circ g_1)$ corresponds to the quasisymmetry

$$g_1^{-1} \circ G^{-1} \circ F \circ f_2 = g_1^{-1} \circ f_1 \circ g_2^{-1} \circ f_2.$$

A decomposition

In fact, every element of $\mathcal{A}$ can be factored uniquely as a product of a "quasisymmetry" and an element of $\mathcal{E}$ (a "bounded univalent function").

Conclusion

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- Conformal field theory can provide algebraic and geometric insight into geometric function theory.
- Example: New model of $T(A)$.
- Two semigroups in function theory are subgroups of a semigroup of non-overlapping mappings, which is naturally related to $T(A)$.