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α-KURAMOTO PARTITIONS FROM THE FRUSTRATED KURAMOTO MODEL GENERALISE EQUITABLE PARTITIONS

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The Kuramoto model describes the collective dynamics of a system of coupled oscillators. An *α-Kuramoto partition* is a graph partition induced by the Kuramoto model, when the oscillators include a phase frustration parameter. We prove that every equitable partition is an α-Kuramoto partition, but that the converse does necessarily not hold. We give an exact characterisation of α-Kuramoto bipartitions.

1. INTRODUCTION

The Kuramoto model is a mathematical model of collective dynamics in a system of coupled oscillators [**9, 4**]. It has been applied in many contexts to describe synchronisation phenomena: *e.g.*, in engineering, for superconducting Josephson junctions, and in biology, for congregations of synchronously flashing fireflies [**1**]; the model has also been proposed to simulate neuronal synchronisation in vision, memory, and other phenomena of the brain [**2**].

We consider a twofold generalisation of the Kuramoto model: firstly, while in the standard case every oscillator is coupled with all the others, we associate the oscillators to the vertices of a graph, as in [**8**]; secondly, with the purpose of including an effect of the graph structure on the dynamics, we take into account a phase frustration parameter, as in [**5**].

Let $G = (V, E)$ be a connected graph on n vertices without self-loops or multiple edges. Let $A(G)$ be the adjacency matrix of G. For each vertex $i \in V(G)$,

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we shall study the following equation, which defines a *frustrated Kuramoto model* (or, equivalently, *frustrated rotator model*) as introduced in [**7**]:

$$\theta_i'(t) := \omega + \lambda \sum_{j=1}^{n} A_{i,j} \sin(\theta_j(t) - \theta_i(t) - \alpha), \qquad t \geq 0, \tag{1}$$

where $\alpha \in [0, \pi/2)$ is a fixed, but arbitrary, phase frustration parameter; this is chosen to be equal for every $i \in V(G)$. The parameter ω is the natural frequency and $\lambda > 0$ is the strength of the interaction, when looking at the system as a set of oscillators. We set $\omega = 0$ and $\lambda = 1$; note that for our purposes, there is no real loss of generality in doing so. By taking $\alpha = 0$, we obtain the standard Kuramoto model [**4**]. We shall consider the case $\alpha \in (0, \pi/2)$.

Let $\theta_i(t)$ be a global smooth solution to the frustrated Kuramoto model for a vertex $i \in V(G)$. There are two natural notions of phase synchronisation:

- Two vertices $i, j \in V(G)$ are said to exhibit *phase synchronisation* when $\theta_i(t) = \theta_j(t)$, for every $t \geq 0$;
- Two vertices $i, j \in V(G)$ are said to exhibit *asymptotic phase synchronisation* when $\lim_{t\to\infty}(\theta_i(t) - \theta_j(t)) = 0$.

We study graph partitions suggested by these notions. Section 2 contains the definition of an α-Kuramoto partition. We show that such partitions are a generalisation of equitable partitions. In Section 3, we exactly characterise α-Kuramoto bipartitions. We also illustrate our results with several examples, revisiting and extending the mathematical analysis done in [**5**]. Some open problems are posed in Section 4.

As a remark, notice that in the standard Kuramoto model ($alpha = 0$) the oscillators can have different natural (or internal) frequencies. Without this the synchronization is reached for any value of $lambda > 0$; instead in the standard Kuramoto model there is a phase transition towards synchronization after some critical coupling, which is related to the natural frequencies distribution.

1. α-KURAMOTO PARTITIONS

Phase synchronisation in the frustrated Kuramoto model induces a partition of a graph. Such a partition will be called an α-Kuramoto partition (see the formal definition below). The α-Kuramoto partitions are closely related to equitable partitions, a well-known object of study in graph theory [**3**]. Theorem 1 states that every equitable partition corresponds to phase synchronisation in the frustrated Kuramoto model. However, not every partition induced by phase synchronisation in the model is an equitable partition.

A *partition* of a graph G is a partition of $V(G)$, *i.e.*, a division of $V(G)$ into disjoint, non-empty sets that cover $V(G)$. The sets of a partition are called *parts*.

Definition 1 (Equitable partition)**.** *A partition $\mathcal{S}$ of a graph G with parts $S_1, ..., S_k$ is* equitable *if the number of neighbours in S_j of a vertex $v \in S_i$ depends only on the choice of the parts S_i and S_j. In this case, the number of neighbours in S_j of any vertex in S_i is denoted γ_{ij}.*

The frustrated Kuramoto model as defined in (1) is naturally associated to a partition as follows:

Definition 2 (α-Kuramoto partition)**.** *Fix $\alpha \in (0, \pi/2)$. A partition $\mathcal{S} = S_1 \cup S_2 \cup \ldots \cup S_k$ of a graph G is called an* α-Kuramoto partition *if there is an initial condition*

$$\theta_j(0) = x_j, \qquad j = 1, \ldots, n, \tag{2}$$

such that the solution to (1) with initial condition (2) has the following properties:

1. *if $i, j \in S_l$ for some l, then $\theta_i(t) = \theta_j(t)$ for all $t \geq 0$;*
2. *if $i \in S_p, j \in S_q$ for distinct indices p, q, then as functions on $[0, \infty)$, we have $\theta_i \neq \theta_j$.*

Example 1. *For each $\alpha \in (0, \pi/2)$, the graph in Figure 1 has an α-Kuramoto partition with two parts: the vertices $\{1, ..., 6\}$ are in one part (black); the vertices $\{7, ..., 10\}$ are in the other part (white).*

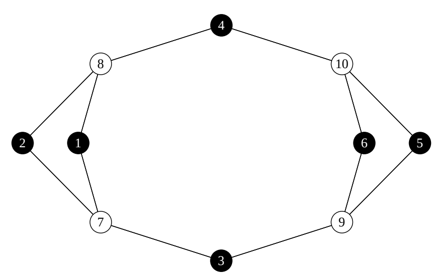

Figure 1: An α-Kuramoto partition with two parts.

Remark 1. A fundamental property of the standard Kuramoto model ($\alpha = 0$) is that $\theta_i(t) = \theta_j(t)$, for all $t \geq 0$ and every two vertices i and j. It is then crucial that $\alpha > 0$, in order to define an α-Kuramoto partition.

The next result establishes a connection between equitable partitions and α-Kuramoto partitions. In particular, it states that every equitable partition is also an α-Kuramoto partition.

Theorem 1. *Let G be a connected graph on n vertices. Suppose that $\mathcal{S}$ is an equitable partition of G. Then for any $\alpha \in (0, \pi/2)$, $\mathcal{S}$ is an α-Kuramoto partition of G.*

Proof. Since $\mathcal{S} = S_1 \cup S_2 \cup \ldots \cup S_k$ is an equitable partition, there are nonnegative integers $\gamma_{i,j}$, with $i, j = 1, \ldots, k$, such that for each pair of indices i, j between 1 and k, any vertex $p \in S_i$, has precisely $\gamma_{i,j}$ neighbours in S_j. Given scalars $c_1 < c_2 < \ldots < c_k$, we consider the following system of differential equations

$$f_i'(t) = \sum_{j=1}^{k} \gamma_{i,j} \sin(f_j(t) - f_i(t) - \alpha), \qquad i = 1, \ldots, k, \tag{3}$$

with initial condition $f_j(0) = c_j$, for $j = 1, \ldots, k$. Observe that the right side of (3) has continuous partial derivatives with respect to each f_l. Hence by Picard's existence theorem [**6**], there is a (unique) solution to this equation with the given initial conditions.

Next, we consider the system (1) with the following initial conditions: for each $i = 1, \ldots, k$ and each $p \in S_i$, we take $\theta_p(0) = c_i$. We construct a solution to this system as follows: for each $p = 1, \ldots, n$, with $p \in S_i$, set $\theta_p(t) = f_i(t)$. It is readily verified that this is yields a solution to (1) with the given initial conditions, and again using Picard's theorem, we observe that such a solution is unique. Evidently, for this solution, we have $\theta_p = \theta_q$ if $p, q \in S_l$, for some l, and $\theta_p \neq \theta_q$ if $p \in S_l, q \in S_m$ and $l \neq m$. □

An *automorphism* of a graph G is a permutation π of $V(G)$ with the property that, for any two vertices $i, j \in V(G)$, we have $\{i, j\} \in E(G)$ if and only if $\{\pi(i), \pi(j)\} \in E(G)$. Two vertices are *symmetric* if there exists an automorphism which maps one vertex to the other. The equivalence classes consisting of symmetric vertices are called the *orbits* of the graph by π.

Definition 3 (Orbit partition)**.** *A partition $\mathcal{S}$ of a graph G with parts $S_1, ..., S_k$ is an* orbit partition *if there is an automorphism of G with orbits $S_1, ..., S_k$.*

The following is an easy fact:

Proposition 1. *An orbit partition is an equitable partition. The converse is not necessarily true.*

Because of this, we have the following intuitive fact.

Proposition 2. *Not every α-Kuramoto partition is an orbit partition.*

The proof is by counterexample. The graph in Figure 1 does not have an automorphism ϕ such that $\phi(1) = 3$, even though vertices 1 and 3 are in the same part of an α-Kuramoto partition.

Remark 2. A graph is said to be *d-regular* if the degree of each vertex is d. Phase synchronisation for a d-regular graph is special. In fact, one α-Kuramoto partition of a d-regular graph is the singleton partition (*i.e.*, it has only one part). To show this, we need to prove that if G is a d-regular graph then $\theta_i(t) = \theta_j(t)$, for all $i, j \in V(G)$. Fix α and let $\theta_i(t) = -d\sin(\alpha)t$, for every $i \in V(G)$. It is now straightforward to verify that $\theta_i(t)$ is a solution to the model.

1. α-KURAMOTO BIPARTITIONS

An *equitable bipartition* (resp. *α-Kuramoto bipartition*) is an equitable partition (α-Kuramoto partition) with exactly two parts. The relationship between equitable bipartitions and α-Kuramoto bipartitions is special. We observe the main property of this relationship in Theorem 2. First, we need a technical lemma whose proof is elementary.

Lemma 1. *Suppose that $f(x)$ is a differentiable function on $[0, \infty)$ and that $f'(x)$ is uniformly continuous on $[0, \infty)$. If $f(x) \to 0$ as $x \to \infty$, then $f'(x) \to 0$ as $x \to \infty$.*

Proof. Suppose to the contrary that $f'(x)$ does not approach 0 as $x \to \infty$. Then there is a sequence x_j diverging to infinity, and $c > 0$ such that $|f'(x_j)| \geq c$, for all $j \in \mathbb{N}$. Without loss of generality we assume that $f'(x_j) \geq c$ for $j \in \mathbb{N}$.

Since f' is uniformly continuous, there is $h > 0$ such that for each $j \in \mathbb{N}$ all $x \in [x_j - h, x_j + h]$ and $f'(x) \geq c/2$. Applying the mean value theorem to f, we see that for each $j \in \mathbb{N}$, there is a $z \in [x_j - h, x_j + h]$ such that

$$f(x_j + h) = f(x_j) + hf'(z) \geq f(x_j) + \frac{ch}{2}.$$

But then for all sufficiently large integers j, we have $f(x_j + h) \geq ch/4$, contrary to hypothesis. The conclusion now follows. □

We are now ready to state the main result of this section.

Theorem 2. *Let G be a connected graph on n vertices. Suppose that $1 \leq k \leq n-1$ and define $S_1 = \{1, \ldots, k\}$, $S_2 = \{k+1, \ldots, n\}$, and $\mathcal{S} = S_1 \cup S_2$. For each $i = 1, \ldots, n$, let $\delta_{i,1}$ be the number of neighbours of vertex i in S_1, and let $\delta_{i,2}$ be the number of neighbours of vertex i in S_2. Suppose that $\alpha \in (0, \pi/2)$ and θ_j ($j = 1, \ldots, n$) is a solution to the frustrated Kuramoto model. Suppose further that:*

- *for all $i, j \in S_1$, we have $\theta_i(t) - \theta_j(t) \to 0$ as $t \to \infty$;*
- *for all $p, q \in S_2$, we have $\theta_p(t) - \theta_q(t) \to 0$ as $t \to \infty$;*
- *for some $i \in S_1$ and $p \in S_2$, $\theta_i(t) - \theta_p(t)$ does not converge to 0 as $t \to \infty$.*

Then one of the following holds:

1. *the partition $\mathcal{S}$ is an equitable partition of G;*
2. *there are scalars μ_1, μ_2, and r such that*

$$-\delta_{i,1} + \mu_1 \delta_{i,2} = r \qquad \textit{for all } i \in S_1,$$

and

$$-\delta_{j,2} + \mu_2 \delta_{j,1} = r \qquad \textit{for all } j \in S_2.$$

Proof. Set $\lambda_1(t) = \frac{1}{k}\sum_{i=1}^{k} \theta_i(t)$ and $\lambda_2(t) = \frac{1}{n-k}\sum_{j=k+1}^{n} \theta_j(t)$. For each $i \in S_1$, let $\epsilon_i = \theta_i - \lambda_1$, and for each $j \in S_2$, let $\epsilon_j = \theta_j - \lambda_2$. From our hypotheses, for each $p = 1, \ldots, n$, $\epsilon_p(t) \to 0$ as $t \to \infty$, while $\lambda_1(t) - \lambda_2(t)$ does not converge to 0 as $t \to \infty$. Observe that each θ_i is differentiable with uniformly continuous derivative, and hence the same is true of λ_1, λ_2, and each of $\epsilon_1, \ldots, \epsilon_n$. In particular, applying Lemma 1 to each ϵ_p, we find that as $t \to \infty$, $\epsilon_p'(t) \to 0$, for $p = 1, \ldots, n$.

For each $i \in S_1$, we have

$$\lambda_1' + \epsilon_i' = \sum_{j \sim i, j \in S_1} \sin(\epsilon_j - \epsilon_i - \alpha) + \sum_{j \sim i, j \in S_2} \sin(\lambda_2 - \lambda_1 - \alpha + \epsilon_j - \epsilon_i)$$

(here we use $j \sim i$ to denote the fact that vertices j and i are adjacent). Now, we rewrite this as

$$\lambda_1' = -\delta_{i,1} \sin(\alpha) + \delta_{i,2} \sin(\lambda_2 - \lambda_1 - \alpha) + \eta_i. \tag{4}$$

Observe that since $\epsilon_p(t), \epsilon_p'(t) \to 0$ as $t \to \infty$, for $p = 1, \ldots, n$, it follows that $\eta_i(t) \to 0$ as $t \to \infty$. Similarly, we find that for each $j \in S_2$,

$$\lambda_2' = \delta_{j,1} \sin(\lambda_1 - \lambda_2 - \alpha) - \delta_{j,2} \sin(\alpha) + \eta_j(t), \tag{5}$$

and that for each $j \in S_2$, $\eta_j(t) \to 0$ as $t \to \infty$.

First, suppose that $\lambda_1(t) - \lambda_2(t)$ does not converge to a constant as $t \to \infty$. Fix two indices $p, q \in S_1$, and note from (4) that

$$(\delta_{p,1} - \delta_{q,2}) \sin(\alpha) + \eta_p - \eta_q = (\delta_{q,2} - \delta_{p,2}) \sin(\lambda_2 - \lambda_1 - \alpha).$$

If $\delta_{q,2} - \delta_{p,2} \neq 0$, we find that as $t \to \infty$, for some $k \in \mathbb{N}$, either

$$\lambda_2 - \lambda_1 \to \arcsin\left(\frac{\delta_{q,1} - \delta_{p,1}}{\delta_{q,2} - \delta_{p,2}} \sin(\alpha)\right) + \alpha + 2\pi k$$

or

$$\lambda_2 - \lambda_1 \to +\alpha + (2k+1)\pi - \arcsin\left(\frac{\delta_{q,1} - \delta_{p,1}}{\delta_{q,2} - \delta_{p,2}} \sin(\alpha)\right),$$

both of which are contrary to our assumption. Hence it must be the case that $\delta_{q,2} = \delta_{p,2}$ which immediately yields that $\delta_{q,1} = \delta_{p,1}$. A similar argument applies for any pair of indices $p, q \in S_2$, and we deduce that $\mathcal{S}$ is an equitable partition.

Now we suppose that for some constant β, $\lambda_1(t) - \lambda_2(t) \to \alpha + \beta$ as $t \to \infty$. Note that then we also have $\lambda_1'(t) - \lambda_2'(t) \to 0$ as $t \to \infty$. Fix indices $p \in S_1, q \in S_2$. From (4) and (5), we find, upon letting $t \to \infty$ that

$$-\delta_{p,1}\sin(\alpha) + \delta_{p,2}\sin(\beta) = -\delta_{q,2}\sin(\alpha) + \delta_{q,1}(-\sin(2\alpha + \beta)).$$

Letting

$$\mu_1 = \frac{\sin(\beta)}{\sin(\alpha)} \text{ and } \mu_2 = \frac{-\sin(2\alpha+\beta)}{\sin(\alpha)}, \tag{6}$$

it now follows that there is a scalar r such that, for all $i \in S_1$ and $j \in S_2$,

$$-\delta_{i,1} + \mu_1\delta_{i,2} = r = -\delta_{j,2} + \mu_2\delta_{j,1}.$$

□

Remark 3. Suppose that condition 2 of Theorem 2 holds. From (6) in the proof of Theorem 2, it is straightforward to verify that $\mu_1 + \mu_2 = -2\cos(\alpha + \beta)$, so that necessarily $|\mu_1 + \mu_2| \le 2$. In particular, for some $k \in N$,

$$\beta = \arccos\left(-\frac{\mu_1+\mu_2}{2}\right) - \alpha + 2\pi k. \tag{7}$$

Observe also that

$$\begin{aligned}\mu_2 &= \frac{-\sin(\alpha)\cos(\alpha+\beta) - \cos(\alpha)\sin(\alpha+\beta)}{\sin(\alpha)}\\ &= \frac{\mu_1+\mu_2}{2} - \cot(\alpha)\sin(\alpha+\beta)\\ &= \frac{\mu_1+\mu_2}{2} - \cot(\alpha)\sqrt{1-\left(\frac{\mu_1+\mu_2}{2}\right)^2}.\end{aligned}$$

It now follows that

$$\alpha = \arctan\left(\frac{\sqrt{4-(\mu_1+\mu_2)^2}}{\mu_1-\mu_2}\right). \tag{8}$$

Moreover, since $\alpha \in (0, \pi/2)$, in fact we must have $|\mu_1 + \mu_2| < 2$ and $\mu_1 > \mu_2$.

Remark 4. Observe that if a graph G happens to satisfy condition 2 of Theorem 2, and if $\mathcal{S}$ is not an equitable partition, then the parameters μ_1, μ_2 and r are readily determined, as follows. Suppose without loss of generality that $|S_1| \ge 2$ and that $\delta_{i,2} \ne \delta_{j,2}$ for some $i, j \in S_1$. Then we find that $\mu_1 = (\delta_{j,1} - \delta_{i,1})/(\delta_{j,2} - \delta_{i,2})$, from which we may find r easily. Since G is connected, $\delta_{j,1}$ is nonzero for some $j \in S_2$, so that μ_2 is then also easily determined. Since μ_1, μ_2 are determined by G, so are α and β. Suppose now that the parameters μ_1, μ_2 so determined are such that

$|\mu_1+\mu_2|<2$ and $\mu_1>\mu_2$, and let α,β be given by (8) and (7), respectively. Select a scalar c, and set initial conditions $\theta_i(0)=c$, for $i\in S_1$, and $\theta_j(0)=c+\alpha+\beta$, for $j\in S_2$. Then the solution to the frustrated Kuramoto model is readily seen to be $\theta_i(t)=c+r\sin(\alpha)t$ $(i\in S_1)$ and $\theta_j(t)=c+r\sin(\alpha)t+\alpha+\beta$ $(j\in S_2)$. In particular, we find that if $\mathcal{S}$ is not an equitable bipartition, but satisfies condition 2 of Theorem 2, then $\mathcal{S}$ is an α–Kuramoto bipartition for α given by (8).

Example 2. Consider the graph G in Figure 2. Let $S_1=\{1,\ldots,5\}$ and $S_2=\{6,\ldots,10\}$. It is readily determined that G satisfies condition 2 of Theorem 2, where our parameters are $\mu_1=1/2$, $\mu_2=1/2$ and $r=0$. We find that $\alpha=\pi/2$ *and* $\beta=\pi/6+2\pi k$ (for some $k\in\mathbb{N}$), and that for an initial condition of the form $\theta_i(0)=c$ $(i\in S_1)$ and $\theta_j(0)=c+2\pi/3+2\pi k$ $(j\in S_2)$ the solution to (1) is constant for all $t\geq 0$. In particular, extending Definition 2 slightly to include the case that $\alpha=\pi/2$, we see that $S_1\cup S_2$ can be thought of as a $\pi/2$–Kuramoto bipartition of G.

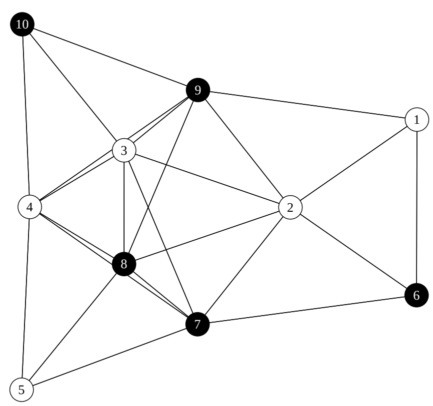

Figure 2: A graph G with a partition of $V(G)$ into $\{1,\ldots,5\}$ and $\{6,\ldots,10\}$ satisfying condition 2 in Theorem 2.

Example 3. Suppose that $p\geq 4$ is even and consider the following adjacency matrix

$$\left[\begin{array}{c|c|c} 0 & \mathbf{1}_p^T & \mathbf{0}_p^T \\ \hline \mathbf{1}_p & \mathbf{0}_{p\times p} & I_p \\ \hline 0_p & I_p & B \end{array}\right],$$

where 0_p and 1_p are the zero and all ones vector in $\mathbb{R}^p$, respectively, $\mathbf{0}_{p\times p}$ and I_p are the zero and identity matrices of order p, respectively, and B is the adjacency matrix of a graph consisting of $p/2$ independent edges. Consider the partition $S=\{1\}\cup\{2,\ldots,2p+1\}$. This partition yields the following parameters: $\delta_{1,1}=0$,

$\delta_{1,2} = p$, $\delta_{i,1} = 1$ ($i = 2, \ldots, p+1$), $\delta_{i,1} = 0$ ($i = p+2, \ldots, 2p+1$), $\delta_{i,2} = 1$ ($i = 2, \ldots, p+1$), $\delta_{i,2} = 2$ ($i = p+2, 2p+1$). It now follows that the graph satisfies condition 2 of Theorem 2, with parameters $\mu_1 = -2/p$, $\mu_2 = -1$, and $r = -2$.

Now, take

$$\alpha = \arctan\left(\frac{\sqrt{3p^2 - 4p - 4}}{p-2}\right) \text{ and } \beta = \arctan\left(\frac{p+2}{2p}\right) - \alpha.$$

Setting up the initial condition

$$\begin{aligned} &\theta_1(0) = 0, \\ &\theta_j(0) = \arccos\left(\frac{p+2}{2p}\right), \qquad j = 2, \ldots, 2p+1, \end{aligned}$$

it follows that the solution to (1) is given by

$$\begin{aligned} &\theta_1(t) = p\sin(\beta)t, \\ &\theta_j(t) = \arccos\left(\frac{p+2}{2p}\right) + p\sin(\beta)t, \qquad j = 2, \ldots, 2p+1. \end{aligned}$$

Example 4. Here we revisit the graph of Figure 1a) in [**5**]. Take $S_1 = \{1\}$ and $S_2 = \{2, \ldots, 7\}$. With this partition, it is straightforward to verify that the graph and partition satisfies condition 2 of Theorem 2, with parameters $\mu_1 = -1/2$, $\mu_2 = -1$, and $r = -2$. It now follows that for $\alpha = \arctan(\sqrt{7})$ and $\beta = \arccos(3/4)$ and initial condition $\theta_1(0) = 0$ and $\theta_j(0) = \alpha + \beta$ ($j = 2, \ldots, 7$), the solution to (1) satisfies $\theta_1(t) = -2t$ and $\theta_j(t) = \alpha + \beta - 2t$ ($j = 2, \ldots, 7$). The figure is drawn below.

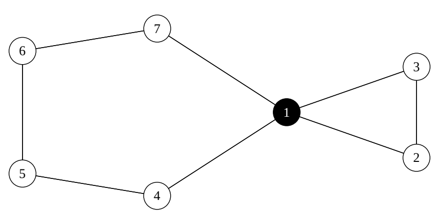

Figure 3: A graph G with a partition of $V(G)$ into $\{1\}$ and $\{2, \ldots, 7\}$ satisfying condition 2 in Theorem 2. This is the graph in Figure 1a of [**5**].

1. OPEN PROBLEMS

Of course the first problem to address, which is central to this paper, is that of establishing a combinatorial characterisation of α-Kuramoto partitions in the general case. Further, we have just touched the surface of asymptotic phase synchronisation in this paper, and more work needs to be done in that direction. Exploring algorithmic applications of this notion is also a potentially fruitful avenue for future research. We conclude the paper with a few specific open problems.

Problem 1. *Can we characterise the degree sequences that satisfy condition 2 of Theorem 2? Specifically, suppose that G is a connected graph and that $S_1 \cup S_2$ is a partition of its vertex set. Let G_{S_1}, G_{S_2} be the subgraphs of G induced by S_1, S_2, respectively. Find necessary and sufficient conditions on the degree sequences of G, G_{S_1} and G_{S_2} in order that the bipartition $S_1 \cup S_2$ satisfies condition 2 of Theorem 2. Observe that the sequences $\delta_{i,1}$ $(i \in S_1)$, $\delta_{j,2}$ $(j \in S_2)$, and $\delta_{p,1} + \delta_{p,2}$ $(p = 1, \ldots, n)$ must all be graphic (*i.e. *must be the degree sequence of some graph).*

Problem 2. *Let G be a connected graph, suppose that the bipartition $S_1 \cup S_2$ satisfies condition 2 of Theorem 2, and let α be given by (8). Determine the initial conditions such that the solutions of the corresponding frustrated Kuramoto model (1) exhibit asymptotic synchronisation with $\lim_{t\to\infty}(\theta_i(t) - \theta_j(t)) = 0$ for all $i, j \in S_1$ and all $i, j \in S_2$.*

Problem 3. *Determine the values of $\alpha \in (0, \pi/2)$ such that for some connected graph G, there is a bipartition of its vertex set as $S_1 \cup S_2$ so that condition 2 of Theorem 2 holds and in addition α is given by (8).*

Problem 4. *We have seen that any regular graph admits a trivial α-Kuramoto partition. Adding pendant vertices or small subgraphs to a regular graph may well change the picture. In the same spirit, it would be interesting to study the behaviour of α-Kuramoto partitions of graph products and other graph operations.*

Problem 5. *What is the family of graphs for which equitable and α-Kuramoto partitions correspond?*

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